There are 10 questions, of which you should attempt 8. Each question carries equal weight. One complete answer is worth more than two partially completed questions. Standard results may be quoted provided they are ***clearly*** stated. All graphs we consider are simple – that is they have no loops or multiple edges – and finite. A graph is r-***regular*** if every vertex has degree r. P_n is the path with n vertices. An ***induced*** subgraph of the graph $G = (V, E)$ is a subgraph of the form $H = (W, F)$ with $W \subset V$ and $F = \{xy \in E \,:\, x, y \in W\}$. Given a graph G its ***clique number***, $\omega(G)$, is the number of vertices in the largest complete subgraph of G. A ***bridge*** in a connected graph is an edge whose removal disconnects the graph.

Question 1. What is the general solution to the recurrence

$$x_n + 3x_{n-1} - 4x_{n-3} = 9?$$

Question 2. How many solutions in integers are there to the system

$$\left.\begin{array}{r} x_1, x_2, \ldots, x_k \geq 1 \\ x_1 + x_2 + \ldots + x_k = m \end{array}\right\}?$$

Prove your answer. How many subsets of $\{1, 2, \ldots, n\}$ contain no subset of the form $\{i, i+1\}$, $1 \leq i \leq n-1$?

Question 3. State and prove Burnside's lemma. [You may assume without proof that if a group G acts on a set X then $|\mathrm{Stab}(x)| \cdot |\mathrm{Orb}(x)| = |G|$, where $\mathrm{Stab}(x)$ is the stabilizer of x and $\mathrm{Orb}(x)$ is its orbit.] How many essentially different ways are there are there to paint the corners of a solid cube with 5 colours?

Question 4. State and prove the Principle of Inclusion-Exclusion. The ***chromatic function*** of a graph G is the function $\chi_G : \mathbb{N} \to \mathbb{N}$ given by

$$\chi_G(k) = |\{c : V(G) \to \{1, 2, \ldots, k\} \,:\, c \text{ is a proper vertex colouring}\}|.$$

Prove using the Principle of Inclusion/Exclusion that the chromatic function is in fact a polynomial in k. Show, moreover, that if G has n vertices and m edges then the leading terms of χ_G are

$$\chi_G(k) = k^n - mk^{n-1} + \ldots$$

Question 5. Let b, k, n, r be positive integers satisfying

$$bk = nr \qquad k < n \qquad b \leq \binom{n}{k}.$$

Prove that there is a family $\mathcal{F}$ of k-subsets of $\{1, 2, \ldots, n\}$ with $|\mathcal{F}| = b$ such that for all $x \in \{1, 2, \ldots, n\}$ the number of sets in $\mathcal{F}$ containing x is r.

Question 6. Suppose that G is a bipartite graph with bipartition (X, Y). Define the ***de£ciency*** of a subset $S \subset X$ to be $\mathrm{def}(S) = |S| - |N(S)|$. Prove that the maximum size of a matching in G is

$$|X| - \max\{\mathrm{def}(S) \,:\, S \subseteq X\}.$$

Question 7. Let G be a graph containing no induced subgraph isomorphic to P_4. Prove that given any ordering of the vertices of G, the greedy algorithm colours G in $\omega(G)$ colours. [Hint: If the greedy algorithm uses k colours, consider the smallest i such that G contains a clique consisting of vertices coloured $i, i+1, i+2 \ldots, k$.]

Question 8. State and prove Euler's Formula concerning planar graphs. The ***girth*** of a graph G is the length of the shortest cycle in G. Prove that if G is planar and bridgeless with n vertices, m edges, and girth g, then

$$m \leq \frac{g}{g-2}(n-2).$$

Question 9. Let G be a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$. The ***adjacency matrix*** of G is the $n \times n$ matrix $A = [a_{ij}]$ with

$$a_{ij} = \begin{cases} 1 & v_i \text{ is adjacent to } v_j \text{ in } G \\ 0 & \text{otherwise} \end{cases}.$$

Prove that the ij^{th} entry of A^k is the number of walks of length k in G from v_i to v_j.

Question 10. Let G be a connected bipartite graph which is k-regular for some $k \geq 2$. Prove that G is bridgeless.