There are 10 questions, of‘ which you should attempt 8. Each question carries equal weight. Standard results may be quoted provided they are *clearly* stated.

We write $[n]$ for the set $\{1, 2, \ldots, n\}$. All graphs we consider are simple – that is they have no loops or multiple edges – and finite. We write $n(G)$ for the number of vertices of G, and $\chi(G)$ for the chromatic number of G (the minimum number of colours required to properly colour the vertices of G). The *chromatic polynomial* of G is the function $\chi_G : \mathbb{N} \to \mathbb{N}$ with $\chi_G(k)$ being the number of proper colourings $c : V(G) \to \{1, 2, \ldots, k\}$.

Question 1.

a. Find the general solution to the following recurrence

$$a_n + 2a_{n-1} - 15a_{n-2} = 3^n.$$

b. A permutation π of $[n]$ is called *connected* if there does not exist any $i < n$ for which π maps $[i]$ to $[i]$. Let c_n be the number of connected permutations of $[n]$. Prove that

$$\sum_{i=1}^{n} c_i (n-i)! = n!.$$

Question 2. Let $\mathcal{B} = \{B_1, B_2, \ldots, B_b\}$ be a family of subsets (called blocks) of $X = [v]$. Further, assume that each unordered pair $\{j, m\} \subset X$ occurs in exactly $\lambda > 0$ blocks. Prove that if each $i \in X$ belongs to strictly more than λ of the blocks then $b \geq v$.

Question 3.

a. State and prove the Principle of Inclusion/Exclusion.

b. A *derangement* is a permutation π with the property that $\pi(i) \neq i$ for all i. Prove that the number of derangements of $[n]$ is the nearest integer to $n!/e$.

Question 4.

a. State Hall's theorem concerning systems of distinct representatives.

b. A *positional game* consists of a set X of positions and a set $\mathcal{W} \subset \mathcal{P}(X)$ of winning sets of positions. [For instance in Tic-Tac-Toe the positions are $X = \{1, 2, 3\}^2$ and the winning sets are those which contain a line.] Two players alternately select positions from X until one player's set of selected positions is in $\mathcal{W}$. (No position can be selected twice.) Suppose that $|W| \geq a$ for all $W \in \mathcal{W}$, and no $x \in X$ belongs to more than b winning sets. Prove that the second player can force a draw if $a \geq 2b$. [Hint: consider the bipartite graph with vertex classes X and two disjoint copies of $\mathcal{W}$. Join $x \in X$ to $W \in \mathcal{W}$ whenever $x \in W$.]

Question 5.

a. State and prove Burnside's lemma. [You may assume without proof that if a group G acts on a set X then $|\mathrm{Stab}(x)| \cdot |\mathrm{Orb}(x)| = |G|$, where $\mathrm{Stab}(x)$ is the stabilizer of x and $\mathrm{Orb}(x)$ is its orbit.]

b. How many different necklaces can be made from n beads each of which is black or white? Two necklaces are considered the same if they only differ by a rotation, but not if they differ by a reflection.

Question 6.
a. Suppose that G is a k-regular graph with $k \geq 1$, having n vertices. Prove that

$$\alpha(G) \leq n/2.$$

b. Suppose that T is a tree having n vertices. Prove that $\alpha(T) \geq n/2$, with equality if and only if T has a perfect matching.

Question 7. Suppose that F, G, H are graphs with $F = G \cup H$ and moreover $G \cap H$ is a clique. Prove that

$$\chi(F; k) = \frac{\chi(G; k)\chi(H; k)}{\chi(G \cap H; k)}.$$

Is the same statement true when $G \cap H$ is not a clique? Give a proof or counter-example.

Question 8. State and prove Turán's Theorem concerning graphs not containing a copy of K_r.

Question 9. Recall that a *block* of a graph G is a maximal connected subgraph of G that has no cut-vertex. [Equivalently the blocks of G are its isolated vertices, its cut-edges, and its maximal 2-edge-connected subgraphs.] Suppose that G is a connected graph containing no even cycles. Prove that every block of G is an edge or an odd cycle.

Question 10. Define $R(s, t)$ to be the minimum value of n such that every (non-necessarily proper) edge colouring of K_n with the colours red and blue contains a monochromatic red K_s or a monochromatic blue K_t. Prove that for all $s, t > 2$,

$$R(s, t) \leq R(s-1, t) + R(s, t-1)$$

and deduce that for all such s, t

$$R(s, t) \leq \binom{s+t-2}{s-1}.$$