Do eight of the ten questions. If you work more than the required number of problems, make sure that you clearly mark which problems you want to have counted. If you have doubts about the wording of a problem or about what results may be assumed without proof, please ask for clarification. In no case should you interpret a problem in such a way that it becomes trivial.

All graphs we consider are simple – that is they have no loops or multiple edges – and finite. A *cubic* graph is one that is 3-regular.

Question 1.

a. State the principle of inclusion/exclusion.

b. Prove that the number of partitions of an n-element set into k non-empty parts satisfies

$$S(n,k) = \frac{1}{k!}\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-i)^n.$$

Question 2.

a. State and prove Burnside's lemma concerning the number of orbits of a group action.

b. How many ways are there to colour the squares in a Tic-Tac-Toe grid with the colours red, white, and blue, if two colourings which differ by a rotation or a reflection are considered the same?

Question 3. Solve the recurrence relation

$$\left.\begin{array}{r} a_n = 3a_{n-2} - 2a_{n-3} \\ a_0 = 3, a_1 = 1, a_2 = 8 \end{array}\right\}$$

Question 4. Give combinatorial proofs of the folowing facts. [Other proof techniques will receive little credit.]

a. $s(n+1,k) = -ns(n,k) - s(n,k-1)$, where $s(n,k)$ is the Stirling number of the first kind, so $(-1)^{n-k}s(n,k)$ is the number of permutations of $\{1,2,\ldots,n\}$ with k cycles.

b. $\displaystyle\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}.$

c. $d_n = (n-1)(d_{n-1} + d_{n-2})$, where d_n is the number of derangements of $\{1,2,\ldots,n\}$.

Question 5.

a. State and prove the Hamming (or "sphere packing") bound on the number of words in a q-ary code with length n and minimum distance at least d.

b. Prove that for binary codes the Hamming bound is always at least as strong as the Singleton bound.

Question 6. Let $A_1, A_2, \ldots, A_n$ be finite sets, and $d_1, d_2, \ldots, d_n$ non-negative integers. Prove that there are disjoint subsets $D_i \subset A_i$ with $|D_i| = d_i$ if and only if for all $I \subset \{1,2,\ldots,n\}$ we have

$$\left|\bigcup_{i\in I} A_i\right| \geq \sum_{i\in I} d_i.$$

Question 7.

a. Prove that every cubic 3 edge-connected graph is 3-connected.

b. Let us define a *snarl* as a graph that can be obtained from K_4 by repeated applications of the operation of subdividing two edges and joining the two new vertices. Show that every snarl is 3-connected. (In fact it is true that every 3-connected cubic graph is a snarl.)

Question 8. Prove that every graph has a bipartition $V(G) = X \cup Y$ with the property that $e(X,Y) \geq e(G)/2$. Further, show that if G is cubic then we can achieve $e(X,Y) \geq n(G) = 2e(G)/3$.

Question 9.

a. State the chromatic recurrence satisfied by the chromatic polynomial $\chi(G;k)$.

b. Prove the for any graph G the number of acyclic orientations of G is $\chi(G;-1)$. [An *orientation* of a graph G is an assignment of a direction to every edge of G. Such an orientation is *acyclic* if the resulting directed graph contains no directed cycles.]

Question 10.

a. State Turán's theorem concerning the maximum number of edges in a graph on n vertices not containing a K_r.

b. Prove that if G is a graph with $n \geq r+1$ vertices and $t_{r-1}(n)+1$ edges then for every n' with $r \leq n' \leq n$ there is a subgraph H of G with n' vertices and at least $t_{r-1}(n')+1$ edges. [Hint: consider a vertex in G of minimum degree.]

c. From the previous part deduce Turán's theorem, and also the stronger fact that such a G contains two K_r subgraphs sharing $r-1$ vertices.