Do seven of the ten questions. If you work more than the required number of problems, make sure that you clearly mark which problems you want to have graded. If you have doubts about the wording of a problem or about what results may be assumed without proof, please ask for clarification. In no case should you interpret a problem in such a way that it becomes trivial.

All graphs we consider are simple – that is they have no loops or multiple edges – and finite. We denote the chromatic number of a graph G by $\chi(G)$.

Question 1. [Note that in both parts we are counting different graphs, not non-isomorphic graphs.]

a. Prove that the number of graphs with vertex set $\{1, 2, \dots, n\}$ is $2^{\binom{n}{2}}$.

b. Prove that the number of graphs with vertex set $\{1, 2, \dots, n\}$ such that all the vertex degrees are even is $2^{\binom{n-1}{2}}$.

Question 2. Let G be a graph that does not contain two disjoint odd cycles. Prove that $\chi(G) \leq 5$. Exhibit such a graph with $\chi(G) = 5$.

Question 3. Let $k \geq 1$. Prove that if G is k-connected and S is a set of k vertices of G then there is a cycle $C \subset G$ with $S \subseteq V(C)$. [Hint: you may assume the Fan Lemma provided you state it clearly.]

Question 4. State and prove Turán's theorem concerning the maximum number of edges in a graph on n vertices not containing a K_r.

Question 5.

a. State Euler's formula concerning the number of faces, edges, and vertices of a plane graph.

b. Prove that a plane graph with n vertices and e edges has $e \leq 3n - 6$ provided $n \geq 3$.

c. An **outerplane** graph is a plane graph in which all the vertices are on the boundary of the outer face. What is the maximum number of edges in an outerplane graph with n vertices? Justify your answer.

Question 6. Prove the following.

a. $F_n^2 - F_{n+1}F_{n-1} = (-1)^n$, where the F_n are the Fibonacci numbers, the solution to the recurrence $F_n = F_{n-1} + F_{n-2}, F_0 = 1, F_1 = 1$.

b. $\displaystyle\sum_{i=0}^{n} i\binom{n}{i} = n2^{n-1}$.

c. The number of solutions in integers to $x_1 + x_2 + \ldots + x_k = n, x_i \geq 1$ is $\displaystyle\binom{n-1}{k-1}$.

Question 7.

a. Find the general form of the solution to the recurrence relation

$$a_n = a_{n-1} + 6a_{n-2} + 12$$

b. Give two sets of initial data, one of which gives a bounded solution, another giving a solution satisfying $\lim |a_n|/p(n) = \infty$ for any polynomial p but $\lim_{n\to\infty} |a_n|/e^n = 0$.

Question 8.

a. State Hall's theorem concerning the existence of a system of distinct representatives for a family $A_1, A_2, \ldots, A_n$ of sets.

b. A **Latin rectangle** is an $m \times n$ matrix with $m \leq n$ whose entries belong to $\{1, 2, \ldots, n\}$ such that each entry appears at most once in any row or column (and hence exactly once in each row). Prove that if $m < n$ then any $m \times n$ Latin rectangle can be extended to an $(m+1) \times n$ Latin rectangle by adding a row.

Question 9.

a. State and prove Burnside's lemma concerning the number of orbits of a group action. [You may assume without proof that the size of the orbit containing a point x is the size of the group divided by the size of the stabilizer of x.]

b. How many essentially different ways are there to paint the surface of a cube with three different colours of paint?

Question 10.

a. State and prove the principle of inclusion/exclusion.

b. Some married couples arrive at a dinner party. How many different ways are there to seat them around a circular table such that no husband and wife sit next to each other?