Do seven of the ten questions. Of these at least three should be from section A and at least three from section B. If you work more than the required number of problems, make sure that you clearly mark which problems you want to have counted. If you have doubts about the wording of a problem or about what results may be assumed without proof, please ask for clarification. In no case should you interpret a problem in such a way that it becomes trivial.

Section A.

Question 1.

a. Simplify $\displaystyle\sum_{k=m}^{n}\binom{k}{m}\binom{n}{k}$.

b. Give, with proof, an expression for the number of surjective functions from a set of size n to one of size k.

Question 2. Let G be a bipartite graph with bipartition A, B that contains a matching from A to B.

a. Prove that for some vertex $a \in A$ it is the case that for all edges $ab \in E(G)$ there is a matching from A to B that contains ab. [Hint: It may be helpful to consider whether or not there is a subset $C \subset A$ having exactly $|C|$ neighbours.]

b. Deduce that if all vertices in A have degree d then the number of matchings from A to B in G is at least $d!$ if $d \leq |A|$ and at least $d(d-1)(d-2)\dots(d-|A|+1)$ if $d > |A|$.

Question 3.

a. State and prove Burnside's lemma. [You may assume without proof that if a group G acts on a set X then $|\mathrm{Stab}(x)| \cdot |\mathrm{Orb}(x)| = |G|$, where $\mathrm{Stab}(x)$ is the stabilizer of x and $\mathrm{Orb}(x)$ is its orbit.]

b. Compute the number of distinguishable ways there are to color the vertices of a solid triangular prism using 3 colors.

Question 4. Let $c_1, c_2, \dots, c_n \geq 0$ be real numbers such that $\sum_1^n c_i = 1$. Prove that the collection $\mathcal{A} = \left\{A \subset [n] \,:\, \sum_{i \in A} c_i > 1/2\right\}$ has size at most 2^{n-1}.

Question 5. Let

$$f(k) = k!\left(1 + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{k!}\right) + 1.$$

Prove that $R(\overbrace{3, 3, \dots, 3}^{k\text{ 3's}}) \leq f(k)$, or in other words that if you color the edges of K_n with k colors and $n \geq f(k)$ then the coloring contains a monochromatic triangle.

Section B.

Question 6. Prove that every non-trivial tree contains at least two maximal independent sets, with equality only for stars.

Question 7.

a. State the Max Flow/Min Cut theorem.

b. Using the Max Flow/Min Cut theorem or otherwise prove that given a collection $(A_i)_1^n$ of sets and integers $(d_i)_1^n$ one can find d_i distinct representatives for A_i if and only if for all $S \subset \{1, 2, \ldots, n\}$ we have

$$\left| \bigcup_{i \in S} A_i \right| \geq \sum_{i \in S} d_i.$$

[To be precise we want to find distinct elements x_{ij} for $1 \leq i \leq n$, $1 \leq j \leq d_i$ with $x_{ij} \in A_i$.]

Question 8. State and prove Turán's theorem, including a proof that the extremal graph is unique up to isomorphism.

Question 9. Given an orientation of a graph G (an assignment of a direction to each edge of G) we define the *length* of the orientation to be the length of the longest directed path in G.

a. Prove that if $\chi(G) \leq k$ then G has an orientation of length at most k.

b. Prove the converse: that if G has an orientation of length at most k then $\chi(G) \leq k$.

Question 10. Suppose that $G = G(n)$ is a graph such that no two vertices of G are joined by 3 internally vertex disjoint paths. Prove that $e(G) \leq \frac{3}{2}(n-1)$. [Hint: Consider the block cut-vertex graph of G.]