Do seven of the ten questions. Of these at least three should be from section A and at least three from section B. If you work more than the required number of problems, make sure that you clearly mark which problems you want to have counted. If you have doubts about the wording of a problem or about what results may be assumed without proof, please ask for clarification. In no case should you interpret a problem in such a way that it becomes trivial.

Section A.

Question 1.

a. Find a closed form for the sum $\sum_{i=0}^{n} \binom{i}{2}\binom{n}{i}$.

b. How many subsets of $\{1, 2, \ldots, n\}$ of size k contain no pair of elements i, j with $|i - j| \le 3$?

Question 2. A class of n students walk to the park one day in single file. On the way back the same students want to walk in single file again, but they want to walk in an order such that no one sees the same person in front of them as they did on the way there. In how many ways can the students be lined up satisfying this constraint?

Question 3. State and prove Sperner's lemma concerning the largest size of an antichain in the power set of $\{1, 2, \ldots, n\}$.

Question 4. Let A be an $n \times n$ matrix with non-negative integer entries. If every row and column of A sums to k prove that A is the sum of k permutation matrices.

Question 5.

a. State and prove Burnside's lemma concerning the number of orbits of a group action. [You may assume without proof that if a group G acts on a set X then $|\mathrm{Stab}(x)| \cdot |\mathrm{Orb}(x)| = |G|$, where $\mathrm{Stab}(x)$ is the stabilizer of $x \in X$ and $\mathrm{Orb}(x)$ is its orbit.]

b. How many essentially different ways are there to color the edges of an regular octahedron with two colors? [Octahedral dice are available on request from the proctor.]

Section B.

Question 6. Consider a family $\mathcal{A} = \{A_1, A_2, \ldots, A_n\}$ of distinct subsets of some n-set X. Define a graph G on $\mathcal{A}$ with an edge between A_i and A_j if $|A_i \triangle A_j| = 1$. If $A_i \triangle A_j = \{x\}$ then we label the edge A_iA_j with x. Prove that there is a forest $F \subset G$ whose edges include all the labels used on edges of G. Deduce that there exists some $x \in X$ for which all the sets $A_1 \setminus \{x\}, A_2 \setminus \{x\}, \ldots, A_n \setminus \{x\}$ are distinct.

Question 7. Prove that a tree T having $2k$ endvertices contains k edge-disjoint paths joining all the endvertices in pairs. Deduce if x is a vertex in a graph G with degree $2k$ and x is not a cutvertex of G then x is contained in k edge-disjoint cycles.

Question 8. Let x be a vertex of a graph G. For $r \geq 0$ define

$$G_r = G\left[\{y \in V(G) \,:\, d_G(x,y) = r\}\right].$$

(In other words G_r is the subgraph of G induced by the vertices at distance r from x.) Prove that

$$\chi(G) \leq \max_r \left(\chi(G_r) + \chi(G_{r+1})\right).$$

Question 9. Prove that a tree T has a perfect matching if and only if the number of odd components in $G - v$ is 1 for every vertex v of G.

Question 10. In this question we consider directed graphs (*digraphs*) with no loops and no multiple edges (but we do allow both $\overrightarrow{xy}$ and $\overrightarrow{yx}$). A *monotone tournament* is an orientation of a complete graph in which (for some ordering of the vertices) the ordering on the edges is from the smaller to the larger vertex. A *complete digraph* is a digraph in which both $\overrightarrow{xy}$ and $\overrightarrow{yx}$ are edges for every x, y in its vertex set.

Given $m \geq 1$ prove that if N is sufficiently large then every digraph on N vertices contains a subset of size m which induces either an empty digraph, a complete digraph, or a monotone tournament.