Do seven of the ten questions. Of these at least three should be from section A and at least three from section B. If you work more than the required number of problems, make sure that you clearly mark which problems you want to have counted. If you have doubts about the wording of a problem or about what results may be assumed without proof, please ask for clarification. In no case should you interpret a problem in such a way that it becomes trivial.

Given a graph G we write $\chi(G)$ for its chromatic number, $\chi'(G)$ for the edge chromatic number, $\omega(G)$ for the clique number, $\alpha'(G)$ for the size of a maximum matching, $e(G)$ for the number of edges, and $\Delta(G)$ for the maximum degree.

Section A.

Question 1.

a. How many pairs (A, B) are there with $A, B \subseteq [n]$ and $A \subset B$? (The inclusion is required to be strict.)

b. Find a closed form for $\displaystyle\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k}$.

Question 2.

a. Let d_n be the number of derangements of $[n]$. Prove that d_n is odd if and only if n is even. (By convention we define $d_0 = 1$.)

b. Suppose that $A = (a_{ij})$ is an $n \times n$ matrix with zeroes on the main diagonal such that $a_{ij} = \pm 1$ for all $i \neq j$. Prove that if n is even then $\det(A) \neq 0$.

Question 3.

a. State and prove Burnside's lemma concerning the number of orbits of a group action. [You may assume without proof that if a group G acts on a set X then $|\mathrm{Stab}(x)| \cdot |\mathrm{Orb}(x)| = |G|$, where $\mathrm{Stab}(x)$ is the stabilizer of $x \in X$ and $\mathrm{Orb}(x)$ is its orbit.]

b. How many different ways are there to color the edges of K_4 with two colors, red and blue? Two colorings are the same if some permutation of the vertices takes one to the other. [One can think of this problem as counting the number of isomorphism classes of graphs on four vertices.]

Question 4. A collection $S_1, S_2, \ldots, S_m$ of subsets of $[n]$ is called *determining* if for all $T \subseteq [n]$ the elements of T can be deduced from the sequence $|T \cap S_1|, |T \cap S_2|, \ldots, |T \cap S_m|$. Show that any determining family has $m \geq n/\log_2(n+1)$.

Question 5. Given an integer $n \geq 0$ and a set of (strictly) positive integers A with $|A| = n^2 + 1$ prove that either

- there is a subset $B \subseteq A$ of size n such that for all $x, y \in B$ either x divides y or y divides x, or
- there is a subset $C \subseteq A$ of size n such that for no pair of distinct $x, y \in C$ do we have x dividing y.

Section B.

Question 6. Prove the König-Egerváry Theorem from the vertex version of Menger's Theorem.

Question 7.
a. Prove that every tree has at most one perfect matching.
b. Let G be a graph with $2m + 1$ vertices and more than $m \cdot \Delta(G)$ edges. Prove that $\chi'(G) > \Delta(G)$.

Question 8. Prove that $\chi(G) = \omega(G)$ when the complement of G is bipartite.

Question 9. Prove that a set of edges in a connected plane graph G forms a spanning tree of G if and only if the duals of the remaining edges form a spanning tree of the dual graph G^*.

Question 10. [An alternative proof of Turán's theorem.] Let G be a maximal K_r-free graph (maximal in the sense of edges) on at least $r - 1$ vertices.
a. Prove that G contains a K_{r-1}.
b. Show that if A is a clique in G of size $r - 1$ then

$$e(G) \leq e(K_{r-1}) + (n - r + 1)(r - 2) + e(G \setminus A).$$

c. Give a proof of Turán's theorem using the previous part.