Do six of the nine questions. Of these at least one should be from each section. If you work more than the required number of problems, make sure that you clearly mark which problems you want to have counted. If you have doubts about the wording of a problem or about what results may be assumed without proof, please ask for clarification. In no case should you interpret a problem in such a way that it becomes trivial. We write $[n]$ for $\{1, 2, \ldots, n\}$.

Section A

Question 1. A *multiset* is like a set except that it can contain an element multiple times. For instance $\{1, 2, 2, 3, 4, 4, 4, 7, 8\}$ contains 1 once, 2 twice, and 4 thrice. The *support* of a multiset A is the set of all elements that are contained in A at least once. The *size* of a multiset A is the total number of elements of A, including repetition (so the size of $\{1, 2, 2, 3, 4, 4, 4, 7, 8\}$ is 9).
a) How many multisets of size k are there with support contained in $[n]$?
b) How many multisets of size k are there with support equal to $[n]$?
c) How many multisets of size k are there with support contained in $[n]$ for which each element is repeated at most twice? [Hint: Inclusion-exclusion.]

Question 2. A *run* in a permutation $\sigma = \sigma_1\sigma_2\ldots\sigma_n$ is a maximal interval $[i, j]$ such that $\sigma_i\sigma_{i+1}\ldots\sigma_j$ is increasing. Thus the permutation $532846791 = 5|3|28|4679|1$ has 5 runs. The Eulerian number $A(n, k)$ is the number of permutations of $[n]$ having exactly k runs.
a) A *descent* in σ is a pair $\{i, i+1\}$ such that $\sigma_i > \sigma_{i+1}$. Find a simple relationship between the number of descents in σ and the number of runs in σ.
b) By considering the reversals of permutations, prove that the total number of runs in all permutations of $[n]$ is $(n + 1)!/2$; i.e.,

$$\sum_{k=1}^{n} kA(n, k) = (n + 1)!/2.$$

c) Prove that for all $n \geq 1$ we have $A(n, 2) = 2^n - n - 1$.

Question 3.
a) Suppose that sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ have associated exponential generating functions $f(x)$ and $g(x)$ respectively. Prove that $f(x)g(x)$ is the exponential generating function of the sequence $(c_n)_{n\geq 0}$ defined by

$$c_n = \sum_{k=0}^{n} \binom{n}{i} a_i b_{n-i}.$$

b) The Bernoulli numbers $(B_n)_{n\geq 0}$ are, by definition, the coefficients appearing in the expansion

$$\frac{x}{e^x - 1} = \sum_{n\geq 0} B_n \frac{x^n}{n!}.$$

By looking at $x(e^{mx} - 1)/(e^x - 1)$ in two ways, prove that $S_r(N)$, the sum of the first N r^{th} powers is given by

$$S_r(N) = \frac{1}{r + 1} \sum_{i=1}^{r+1} B_{r+1-i} \binom{r + 1}{i} (N + 1)^i$$

Section B

Question 4. Two permutations $\sigma_1\sigma_2\ldots\sigma_n$ and $\tau_1\tau_2\ldots\tau_n$ of $[n]$ *intersect* if for some i we have $\sigma_i = \tau_i$ (i.e., if they agree in some position). A family of permutations of $[n]$ is called *intersecting* if every pair of permutations from the family intersects. Prove that the maximum size of an intersecting family of permutations of $[n]$ is $(n-1)!$.

Question 5.
a) Suppose that $A : \mathbb{F}_2^m \to \mathbb{F}_2^r$ is a linear map. Define a code by

$$\mathcal{C} = \{(v, Av) \,:\, v \in \mathbb{F}_2^m\}.$$

Find the length and dimension of $\mathcal{C}$, and prove that its minimum distance is

$$\min_{\substack{v\in\mathbb{F}_2^m \\ v\neq\mathbf{0}}} \big(\mathrm{wt}(v) + \mathrm{wt}(Av)\big).$$

b) Suppose that d is odd. Prove that if a binary $[n, k, d]$ linear code exists then a binary $[n + 1, k, d + 1]$ linear code also exists.

Question 6. Fix $m > 0$. A linear q-ary code C is called a *Hamming code* if it has a parity-check matrix H, where H is an $m \times n$ matrix whose columns consist of one nonzero vector from each of the 1-dimensional subspaces of $\mathbb{F}_q^m$. Determine the parameters n, k, d such that C is a linear $[n, k, d]$ code. Show that C is perfect.

Section C

Question 7. Let G be a bipartite graph with partite sets of size $n/2$ and minimum degree at least $n/4+1$. Prove that G is Hamiltonian. Show that this is sharp whenever 4 divides n by constructing for each such n an n-vertex non-Hamiltonian bipartite graph with minimum degree $n/4$. [Hint: Consider a longest path.]

Question 8. Prove that every plane n-vertex graph isomorphic to its dual has $2n - 2$ edges. For each $n \geq 4$, construct an n-vertex (simple) plane graph isomorphic to its dual.

Question 9. Given a set of lines in the plane with no three meeting at a point, form a graph G whose vertices are the intersections of the lines, with two vertices adjacent if they appear consecutively on one of the lines. (An example of a set of lines and the corresponding graph are shown below.) Prove that the chromatic number $\chi(G)$ is at most 3.