Do seven of the ten questions. Of these at least three should be from section A and at least three from section B. If you work more than the required number of problems, make sure that you clearly mark which problems you want to have counted. If you have doubts about the wording of a problem or about what results may be assumed without proof, please ask for clarification. In no case should you interpret a problem in such a way that it becomes trivial.

We say that $f(n) = O(g(n))$ if there exists a constant $C > 0$ and an integer $N \geq 1$ such that for all $n \geq N$ we have $|f(n)| \leq C|g(n)|$.

Section A.

Question 1.

a. Find a closed form for the sum $\sum_{i=0}^{n} \binom{n}{i}^2$.

b. Give the definition of the (signed) Stirling numbers of the first kind (those counting certain permutations), and prove that they satisfy the recurrence

$$s(n+1, k) = -ns(n, k) + s(n, k-1).$$

Question 2.

a. State and prove the principle of inclusion/exclusion.

b. In the village of Inaugura there are $2n$ people, made up of n married couples, who wish to speak at a certain town meeting. In Inaugura it is considered a dreadful *faux pas* for the two members of a couple to speak consecutively. In how many possible orders can all $2n$ people speak, respecting this condition?

Question 3.

a. State Hall's theorem.

b. We call a bipartite graph with bipartition (X, Y) *biregular* if all the vertices in X have the same degree and also all the vertices in Y have the same degree (not necessarily the same as those in X). Prove that a biregular bipartite graph has a matching saturating the smaller side.

c. Prove that the partially ordered set $\mathcal{P}(n) = \{A : A \subseteq [n]\}$ (ordered by inclusion) can be partitioned into $\binom{n}{\lfloor n/2 \rfloor}$ subsets, each of which is a chain.

Question 4. Find the general solution of the recurrence relation

$$a_n = a_{n-1} + 8a_{n-2} - 12a_{n-3} + 8.$$

What is the smallest λ for which the solution is guaranteed to be $O(\lambda^n)$?

Question 5.

a. Define the *generator matrix* and *parity check matrix* of a linear code C over $\mathbb{F}_2$ (the field with two elements).

b. Let G, H be matrices with entries in $\mathbb{F}_2$ where G is $k \times n$ and H is $(n-k) \times n$. Suppose that both have full rank. Prove that there exists some code C such that G is its generator matrix and H is its parity check matrix if and only if $GH^T = 0$.

c. Describe the technique of syndrome decoding for a linear code with parity check matrix H.

Section B.

Question 6. [The two parts of this question are not related.]
a. An *outerplanar* graph is a planar graph that has some embedding in the plane for which every vertex is on the outer face. Determine the maximum number of edges in an outerplanar graph having n vertices, using Euler's formula. Show that your bound is sharp with a construction.
b. Prove that a planar graph is bipartite if and only if every face has even length.

Question 7. Prove that every 3-connected graph with at least 6 vertices that contains a subdivision of K_5 also contains a subdivision of $K_{3,3}$.

Question 8. State and prove Turán's Theorem concerning graphs not containing a copy of K_r.

Question 9. Given a graph G with n vertices, let $\overline{G}$ denote the complement of G. By using induction on the number of vertices of G, or otherwise, prove that $\chi(G) + \chi(\overline{G}) \leq n + 1$.

Question 10. Given a graph $G = (V, E)$, the *line graph of* G, denoted $L(G)$, is the graph with vertex set E, in which two vertices e, f are adjacent if and only if the edges e, f are incident in G. A graph H is a line graph if there exists a graph G such that $H = L(G)$. Use Tutte's 1-factor theorem to prove that every connected line graph with an even number of vertices has a perfect matching.