Do six of the nine questions. Of these at least one should be from each section. If you work more than the required number of problems, make sure that you clearly mark which problems you want to have counted. If you have doubts about the wording of a problem or about what results may be assumed without proof, please ask for clarification. In no case should you interpret a problem in such a way that it becomes trivial.

Section I

1. Let a_n be the number of n-letter words that can be formed using the letters x,y, and z such that any nonterminal x has to be immediately followed by a y.
   (a) Find a recurrence relation for a_n and the initial conditions for this recurrence relation.
   (b) Solve the recurrence relation you obtained in part (a) using the characteristic equation method.

2. Evaluate the sum below using generating functions, and then give an inclusion-exclusion proof of the resulting identity.
$$\sum_{k=0}^{m}(-1)^k\binom{n}{k}\binom{n-k}{m-k}$$

3. (a) State and prove Burnside's lemma concerning the number of orbits of a group action. [You may assume without proof that if a group G acts on a set X then $|Stab(x)|\cdot|Orb(x)| = |G|$, where $Stab(x)$ is the stabilizer of $x \in X$ and $Orb(x)$ is its orbit.]
   (b) Find the number of distinguishable colorings of the squares of a 3×3 chessboard with 3 colors if each color is to be used at least once. (Assume that two colorings are the same if they differ by a rotation or a reflection.)

Section II

4. Let $\mathcal{C}$ be a binary linear code with the following parity-check matrix H.
$$H = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$
   (a) Determine the block length, dimension, and the minimum distance of $\mathcal{C}$.
   (b) Determine whether $\mathbf{1}$ (i.e. the all-ones vector) is a codeword in $\mathcal{C}$.
   (c) Determine whether $\mathcal{C}$ is self-dual.

5. (a) What does it mean for a linear code to be *maximum distance separable* (MDS)?
   (b) Prove that every Reed-Solomon code is MDS.

6. Let G be the incidence graph of the Fano plane; G is bipartite and has 14 vertices. Using the properties of the Fano plane (without drawing G), determine the minimum size of a dominating set in G. (A *dominating set* in a graph G is a subset S of the vertices in G such that every vertex v in $V(G) - S$ has a neighbor in S.)

Section III

7. Apply Hall's Theorem to prove that there is a tournament with outdegrees $p_1, p_2, \ldots, p_n$ if and only if for each k, the k smallest of these numbers sum to at least $\binom{k}{2}$, with equality when $k = n$. (*Hint*: Construct an X, Y-bigraph in which Y is the set of pairs from $\{1, 2, \ldots, n\}$ and X consists of p_i copies of i for each $i \in \{1, 2, \ldots, n\}$.)

8. *Alternate proof of Turán's Theorem.* Let $t_r(n)$ denote the maximum number of edges in a simple graph on n vertices with no $(r+1)$-clique.

   (a) Prove that a maximal graph with no $(r+1)$-clique has an r-clique.

   (b) Prove that $t_r(n) = \binom{r}{2} + (n-r)(r-1) + t_r(n-r)$.

   (c) State and prove Turán's Theorem using parts (a) and (b).

9. Given a graph G on n vertices, the j^{th} power G^j of G is the graph with vertex set $V(G)$ and edge set $\{uv : d_G(u,v) \leq j\}$. Prove that if G is k-connected and $jk < n$, then G^j is jk-connected.