Do six of the nine questions. Of these at least one should be from each section. If you work more than the required number of problems, make sure that you clearly mark which problems you want to have counted. If you have doubts about the wording of a problem or about what results may be assumed without proof, please ask for clarification. In no case should you interpret a problem in such a way that it becomes trivial.

We write $[n]$ for $\{1, 2, \ldots, n\}$. If $\mathbb{F}$ is a finite alphabet and $\mathcal{C} \subseteq \mathbb{F}^n$ is a code we say that $\mathcal{C}$ is an (n, M, d) code if $|\mathcal{C}| = M$ and $\mathcal{C}$ has minimum distance d. Similarly we say $\mathcal{C}$ is a an $[n, k, d]$ code if $\mathbb{F}$ is a finite field, $\mathcal{C}$ is a linear subspace of $\mathbb{F}^n$ of dimension k, and the minimum distance of $\mathcal{C}$ is d.

Section A

Question 1.

a) State and prove the exponential formula relating the exponential generating function (egf) enumerating structures made from "connected" substructures, and the egf enumerating the connected structures.

b) What is the exponential generating function for the number of permutations of $[n]$ whose sixth power is the identity?

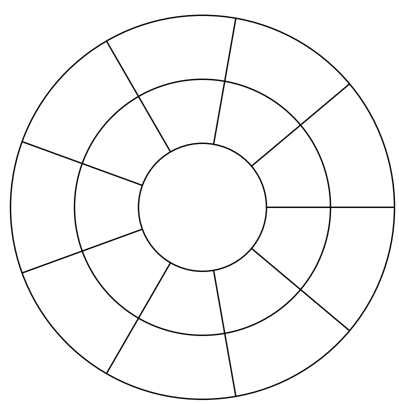
Question 2.

a) What is the general solution to the recurrence

$$a_n = 2a_{n-1} + 8a_{n-2} - 5(3^n)?$$

b) Say that a function $f : \mathbb{N} \to \mathbb{R}$ is *simple* if there is a homogeneous linear recurrence with constant coefficients satisfied by f. Thus for instance $f(n) = n3^n$ is simple since it satisfies $f(n) = 6f(n-1) - 9f(n-2)$. Prove that if $f, g : \mathbb{N} \to \mathbb{R}$ are simple then so is their product fg.

Question 3. How many ways are there to color the regions in the diagram below black or white if two colorings that differ only by a rotation or reflection are considered the same?



Section B

Question 4.

a) State and prove Sperner's theorem concerning the size of a largest antichain in the cube $Q^n = \{0, 1\}^n$.

b) Prove that the number of antichains in Q^n is at most $(n + 1)^{\binom{n}{\lfloor n/2 \rfloor}}$.

Question 5. Consider an (n, M_1, d_1) code $\mathcal{C}_1$ and an (n, M_2, d_2) code $\mathcal{C}_2$ over the same alphabet $\mathbb{F}$, so $\mathcal{C}_1, \mathcal{C}_2 \subseteq \mathbb{F}^n$. Define a new code

$$\mathcal{C} = \{(\mathbf{u}, \mathbf{u}+\mathbf{v}) \,:\, \mathbf{u} \in \mathcal{C}_1, \mathbf{v} \in \mathcal{C}_2\} \subseteq \mathbb{F}^{2n}.$$

Prove that $\mathcal{C}$ is a $(2n, M_1M_2, d)$ code where $d = \min\{2d_1, d_2\}$.

Question 6. Consider a memoryless binary erasure channel with input alphabet $A = \{0, 1\}$, output alphabet $B = \{0, 1, ?\}$, and erasure probability p. [Thus each symbol independently is either left unchanged with probability $1-p$ or replaced with a ? with probability p.] Let $\mathcal{C}$ be the binary $[4, 3, 2]$ code with parity check matrix $\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}^{\mathrm{T}}$. A uniformly chosen codeword of $\mathcal{C}$ is transmitted over the channel and the following decoder $\mathcal{D} : B^4 \to \mathcal{C} \cup \{\texttt{error}\}$ is applied to the received word $\mathbf{y}$:

$$\mathcal{D}(\mathbf{y}) = \begin{cases} \mathbf{c}, & \text{if there exists a unique } \mathbf{c} \in \mathcal{C} \text{ such that } \mathbf{c} \text{ and } \mathbf{y} \text{ agree in every} \\ & \text{place where the entry of } \mathbf{y} \text{ is in } A \\ \texttt{error} & \text{otherwise.} \end{cases}$$

What is the probability that $\mathbf{y}$ is decoded to `error`?

Section C

Question 7.

a) Suppose that the degree sequence of a graph G is $(d_1, d_2, \ldots, d_n)$ where $d_1 \geq d_2 \geq \cdots \geq d_n$. Prove that

$$\chi(G) \leq \max\{\min(i, d_i + 1) \,:\, i \in [n]\}.$$

b) Prove that for any graph G on n vertices we have $\chi(G) + \chi(\overline{G}) \leq n + 1$.

Question 8. Prove that there is a tournament (an orientation of K_n) with outdegrees $d_1 \leq d_2 \leq \cdots \leq d_n$ (where the d_i are non-negative integers) if and only if for all $1 \leq k \leq n$ we have

$$\sum_{1}^{k} d_i \geq \binom{k}{2},$$

with equality when $k = n$. [Hint: apply Hall's theorem to a bipartite graph with $\binom{n}{2}$ vertices in each part.]

Question 9. Let G be a graph containing k edge-disjoint spanning trees. Suppose that $e_1, e_2, \ldots, e_k$ are distinct edges in G. Prove that there are disjoint spanning trees $T_1, T_2, \ldots, T_k$ such that $e_i \in E(T_i)$ for $i \in [k]$.