Do six of the nine questions. Of these at least one should be from each section. If you work more than the required number of problems, make sure that you clearly mark which problems you want to have counted. Standard results may be used without proof provided they are clearly stated, though in no case should you interpret a problem in such a way that it becomes trivial. If you have doubts about the wording of a problem or about what results may be assumed without proof, please ask for clarification.

We write $[n]$ for the set $\{1, 2, \ldots, n\}$. If A is a set then we write $\binom{A}{k}$ for the collection of all k-subsets of A.

Section A

Question 1. The parts of this question are unrelated.

a) Prove that there are $\binom{n-1}{k-1}$ compositions of n having exactly k parts. (A *composition* of n is a sequence $(x_1, x_2, \ldots, x_k)$ of positive integers such that $\sum_i x_i = n$.)

b) Prove that for $x, y \in \mathbb{N}$ and $n \in \mathbb{N}_0$ we have

$$(x+y)^{(n)} = \sum_{r=0}^{n} \binom{n}{r} x^{(r)}\, y^{(n-r)}.$$

Here $x^{(r)} = x(x+1)(x+2)\cdots(x+r-1)$ is the rising factorial.

Question 2.

a) Find the solution to the linear recurrence relation

$$\begin{cases} a_n = 5a_{n-1} - 8a_{n-2} + 4a_{n-3} & n \geq 3 \\ a_0 = -1 \quad a_1 = 1 \quad a_2 = 7 \end{cases}$$

b) Suppose that $(b_n)_{n\geq 0}$ is the solution to a k^{th}-order linear recurrence with constant coefficients, so that for some constants $c_1, c_2, \ldots, c_k$ we have

$$b_n = c_1 b_{n-1} + c_2 b_{n-2} + \ldots c_k b_{n-k}$$

for $n \geq k$. Prove that there exist $A, r \in \mathbb{R}$ and $t \in \mathbb{N}_0$ such that

$$b_n = An^t r^n + e_n,$$

where e_n satisfies

$$\lim_{n\to\infty} \frac{e_n}{n^t r^n} = 0.$$

Question 3.

a) How many subsets of $\binom{[n]}{2}$ consist of exactly k disjoint pairs?

b) Using the Principle of Inclusion/Exclusion, or otherwise, give a formula for the number of permutations of $[n]$ having no 2-cycles.

Section B

Question 4.

a) Define what it means for a matrix G to be a *generator matrix*, and a matrix H to be a *parity check matrix*, for a linear code C over a finite field $\mathbb{F}$.

b) Let G, H be full rank matrices over $\mathbb{F}$, where G is $k \times n$ and H is $(n-k) \times n$. Prove that there exists a code C with generator matrix G and parity check matrix H if and only if $GH^{\mathrm{T}} = 0$.

c) The *binary Hamming code* $\mathrm{Ham}_2(r)$ is a linear code in $\mathbb{F}_2^n$ where $n = 2^r - 1$. It is specified by its parity-check matrix, which is an $r \times n$ matrix having all non-zero vectors in $\mathbb{F}_2^r$ as its columns. Compute, with justification, the length, dimension, and minimum distance of $\mathrm{Ham}_2(r)$.

Question 5. Suppose that $\mathcal{A} \subseteq \mathcal{P}(n)$ is an antichain (so for all $A, B \in \mathcal{A}$ we have $A \subseteq B$ implies $A = B$) such that $|A| \le k$ for all $A \in \mathcal{A}$, for some $k \le n/2$. Prove that

$$|\mathcal{A}| \le \binom{n}{k}.$$

Question 6. Consider a family $\mathcal{I} \subseteq \mathcal{P}(n)$ that is *intersecting* (i.e., $A, B \in \mathcal{I} \implies A \cap B \neq \emptyset$).

a) Prove that $|\mathcal{I}| \le 2^{n-1}$.

b) Prove that there exists $\tilde{\mathcal{I}}$ such that $\mathcal{I} \subseteq \tilde{\mathcal{I}} \subseteq \mathcal{P}(n)$, $\tilde{\mathcal{I}}$ is intersecting, and $|\tilde{\mathcal{I}}| = 2^{n-1}$.

Section C

Question 7. Let G be a graph on $n \ge 4$ vertices not containing C_4 as a subgraph. Prove that $\chi(G) \le \alpha'(G) + 2$. (Here, as usual, $\alpha'(G)$ is the size of a largest matching in G.)

Question 8. Let $G = (V, E)$ be a graph.

a) Prove that one can find a partition $V = U \cup W$ of the vertex set such that $e(U, W) \ge \frac{1}{2} e(G)$.

b) Prove that if G is 3-regular then one can achieve $e(U, W) \ge n = \frac{2}{3} e(G)$.

Question 9. Let X, Y be disjoint sets of vertices in a k-connected graph G and the $w : X \cup Y \to \mathbb{N}_0$ be a weight function satisfying

$$\sum_{x \in X} w(x) = k = \sum_{y \in Y} w(y).$$

Prove that there are k internally vertex disjoint X–Y paths such that exactly $w(v)$ paths have v as an endpoint for each $v \in X \cup Y$.