Do six of the nine questions. Of these at least one should be from each section. If you work more than the required number of problems, make sure that you clearly mark which problems you want to have counted. Standard results may be used without proof provided they are clearly stated, though in no case should you interpret a problem in such a way that it becomes trivial. If you have doubts about the wording of a problem or about what results may be assumed without proof, please ask for clarification.

We write $[n]$ for the set $\{1, 2, \ldots, n\}$.

Section A

Question 1.

a) Prove that for all integers $n, r \geq 0$ we have

$$\sum_{k=0}^{n} \binom{k}{r} = \binom{n+1}{r+1}.$$

b) Prove that for all integers $n, m, r \geq 0$ we have

$$\sum_{k} \binom{n}{k}\binom{m}{r+k} = \binom{n+m}{n+r}.$$

Question 2. A permutation π of $[n]$ is called *connected* if there is no $0 < k < n$ such that $\pi([k]) = [k]$. Denote the number of connected permutations of $[n]$ by c_n. By convention we set $c_0 = 1$.

a) Prove that for $n \geq 1$ we have

$$\sum_{k=1}^{n} c_k (n-k)! = n!.$$

b) Deduce that the generating functions $F(x)$ and $G(x)$, for the sequences $(n!)_{n\geq 1}$ and $(c_n)_{n\geq 1}$ respectively, satisfy

$$F(x) = \frac{1}{2 - G(x)},$$

as formal power series.

Question 3.

a) State and prove Burnside's lemma concerning the number of orbits of a group action. [You may assume without prove that if a group G acts on a set X then for all $x \in X$ we have $|\mathrm{Stab}(x)| \cdot |\mathrm{Orbit}(x)| = |G|$ where $\mathrm{Stab}(x)$ is the stabilizer of x and $\mathrm{Orbit}(x)$ is its orbit.]

b) Find the number of distinguishable necklaces with 6 beads chosen from four types of beads.

Section B

Question 4.

a) State and prove Dilworth's theorem concerning antichains and chain covers in finite posets.

b) Use Dilworth's theorem to prove the König-Egerváry theorem, that in a bipartite graph the size of the largest matching is the same as the size of the smallest vertex cover.

Question 5. Suppose that $\mathcal{C} \subseteq \mathbb{F}_2^n$ is a (binary) linear code of dimension k.

a) Prove that either all the codewords in $\mathcal{C}$ have even weight or the number of even weight codewords is the same as the number of odd weight codewords.

b) Prove that either all the codewords in $\mathcal{C}$ begin with 0 or half begin with 0 and half with 1.

c) Prove that

$$\sum_{\mathbf{x} \in \mathcal{C}} \mathrm{wt}(\mathbf{x}) \leq n 2^{k-1}.$$

Question 6. Let G be a $k \times n$ generator matrix of a linear code $\mathcal{C}$ over a finite field $\mathbb{F}$. Prove that the minimum distance of $\mathcal{C}$ is the largest integer d such that every $k \times (n-d+1)$ sub-matrix of G has full row rank (i.e., rank k).

Section C

Question 7. Let G be a connected graph with at least $k+1$ vertices. Prove that G is k-connected if and only if whenever $\mathrm{dist}(x, y) = 2$ there is a family of k internally vertex-disjoint x, y-paths in G. [You may assume Menger's Theorem provided you state it clearly.]

Question 8. Let G be a connected graph with n vertices. Define a graph $\mathcal{T}(G)$ whose vertices are the spanning trees of G, in which two spanning trees T and T' are adjacent exactly if

$$|E(T) \vartriangle E(T')| = 2$$

(or in other words T and T' share as many edges as it is possible for two distinct spanning trees to share).

a) Prove that $\mathcal{T}(G)$ is connected.

b) Find (with proof) a simple expression for $\mathrm{dist}_{\mathcal{T}(G)}(T, T')$, where T, T' are spanning trees of G.

Question 9. Consider an outerplane drawing D of a graph G. (A graph drawing is called *outerplane* if all the vertices lie on the outer face.)

a) Prove that the graph obtained from the planar dual of D by removing the vertex corresponding to the outer face is a tree.

b) Prove that every 2-connected graph G with an outerplane drawing and $n(G) > 2$ has a vertex of degree 2.