Do six of the nine questions. Of these at least one should be from each section. If you work more than the required number of problems, make sure that you clearly mark which problems you want to have counted. If you have doubts about the wording of a problem or about what results may be assumed without proof, please ask for clarification. In no case should you interpret a problem in such a way that it becomes trivial. We write $[n]$ for the set $\{1, 2, \ldots, n\}$.

Section A

Question 1.

a) Solve the recurrence

$$\begin{cases} a_n = a_{n-1} + 5a_{n-2} + 3a_{n-3}, & n \geq 3 \\ a_0 = 1, \quad a_1 = 2, \quad a_2 = 4 \end{cases}$$

b) Prove that every polynomial satisfies some finite order, linear, homogeneous, constant coefficient recurrence.

Question 2.

a) State and prove the Orbit Counting Lemma (sometimes called Burnside's Lemma), relating the number of orbits in a group action $G \rightarrowtail X$ to the the number of fixed points of the various group elements $g \in G$.

b) A *double domino* is a solid two-sided chip of wood such that each side consists of two squares sharing an edge. Each of the four squares is marked with a number between 0 and $k - 1$, inclusive. Determine the number of distinguishable double dominos.

Question 3. A *ranking* of candidates in an election is an ordering of the candidates that allows ties. Let a_n be the number of possible rankings for an election with n candidates. (Note that $a_2 = 3$ and $a_3 = 13$.) Recall that $S(n, k)$ is the number of (set) partitions of $[n]$ into k parts.

a) Find a simple sum for a_n involving the $S(n, k)$.

b) Prove that $k!S(n, k) = \sum_i (-1)^i \binom{k}{i} (k - i)^n$.

c) Prove that the exponential generating function for $(a_n)_{n \geq 0}$ is $1/(2 - e^x)$.

Section B

Question 4. A *Hadamard matrix* is a matrix $H \in M_{n \times n}(\mathbb{R})$ with ± 1 entries such that $HH^{\mathrm{T}} = nI$.

a) Prove that if H is a Hadamard matrix then the rows of H, together with their negatives, form a (not linear) code over the alphabet $\{\pm 1\}$ of size $2n$, length n, and minimum distance $n/2$.

b) The tensor product of matrices $A \in M_{r \times r}$ and $B \in M_{s \times s}$ is defined to be the $rs \times rs$ matrix

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1r}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{r1}B & a_{r2}B & \cdots & a_{rr}B \end{bmatrix}.$$

Prove that if A and B are Hadamard matrices then so is $A \otimes B$.

c) Deduce that there are Hadamard matrices with $n = 2^\ell$ for all $\ell \in \mathbb{N}$.

Question 5. Prove that if $\mathcal{F}$ is an antichain of subsets of $[n]$ consisting of pairs of complementary sets, then $|\mathcal{F}| \leq 2\binom{n-1}{\lfloor n/2 \rfloor - 1}$.

Question 6. Suppose that in a red/blue coloring of $E(K_n)$, the subgraph formed by the red edges is transitively orientable. In other words, there is some poset P on $[n]$ such that elements $i, j \in [n]$ are comparable in P if and only if they are joined by a red edge in the coloring.

Show that if $n > m^2$, then there is a monochromatic complete subgraph in this coloring with at least $m + 1$ vertices.

Section C

Question 7. Determine all connected graphs G such that the edge chromatic number $\chi'(G)$ is strictly less than the chromatic number $\chi(G)$.

Question 8. Prove that if G is an X, Y-bigraph with $|X| = |Y| = n$, then $\alpha'(G) \geq \min\{2\delta(G), n\}$. [Recall that $\alpha'(G)$ is the size of a largest matching in G.]

Question 9. The graph below are called the *claw* and the *paw* respectively. Prove that a connected graph G that contains a cycle and does not contain either a claw or a paw as an induced subgraph is Hamiltonian.