Do six of the nine questions. Of these at least one should be from each section. If you work more than the required number of problems, make sure that you clearly mark which problems you want to have counted. If you have doubts about the wording of a problem or about what results may be assumed without proof, please ask for clarification. In no case should you interpret a problem in such a way that it becomes trivial. We write $[n]$ for $\{1, 2, \dots, n\}$.

Section A

Question 1. Let a_n denote the number of words of length n over the alphabet $\{A, B, C\}$ that contain no consecutive A's and no consecutive B's.

a) Show that a_n satisfies the recurrence $a_n = 2a_{n-1} + a_{n-2}$ for all $n \geq 2$.

b) Find an explicit expression for a_n by solving the recurrence.

Question 2. Given a fixed $k \in \mathbb{N}$, let f_n denote the number of surjections from $[n]$ to $[k]$. Note that these surjections can be also viewed as words of length n over alphabet $[k]$ with no missing symbols.

a) Compute f_n by first finding the exponential generating function and then extracting the general coefficient.

b) Compute f_n by using the Inclusion-Exclusion principle.

Question 3.

a) State and prove Burnside's lemma concerning the number of orbits of a group action. [You may assume without proof that if a group G acts on a set X then $|\operatorname{Stab}(x)| \cdot |\operatorname{Orb}(x)| = |G|$, where $\operatorname{Stab}(x)$ is the stabilizer of $x \in X$ and $\operatorname{Orb}(x)$ is its orbit.]

b) Let p, q be distinct primes. Compute the number of different necklaces that can be made from pq beads, where each bead is one of k different kinds.

Section B

Question 4.

a) Define what it means for a matrix G to be a *generator matrix*, and a matrix H to be a *parity-check matrix*, for a linear code $\mathcal{C}$ over a finite field $\mathbb{F}$.

b) Let G, H be full rank matrices over $\mathbb{F}$, where G is $k \times n$ and H is $(n-k) \times n$. Prove that there exists a code $\mathcal{C}$ with generator matrix G and parity-check matrix H if and only if $GH^T = 0$.

c) The *binary Hamming code* $\operatorname{Ham}_2(r)$ is a linear code in $\mathbb{F}_2^n$ where $n = 2^r - 1$. It is specified by its parity-check matrix, which is an $r \times n$ matrix having all non-zero vectors in $\mathbb{F}_2^r$ as its columns. Compute, with justification, the length, dimension, and minimum distance of $\operatorname{Ham}_2(r)$.

Question 5. Let $A_1, A_2, \dots, A_n$ be finite sets, and $d_1, d_2, \dots, d_n$ be non-negative integers. Prove that there exist disjoint sets $D_1, D_2, \dots, D_n$, with $D_i \subseteq A_i$ and $|D_i| = d_i$ for all i, if and only if for all $I \subseteq [n]$ we have

$$\left|\bigcup_{i \in I} A_i\right| \geq \sum_{i \in I} d_i.$$

Question 6. Suppose that C_1, C_2 are binary linear codes, where C_i has length n_i, dimension k_i and minimum distance d_i. Consider the vector space V of $n_1 \times n_2$ binary matrices and the subspace C of those matrices where every column is an element of C_1 and every row is an element of C_1. C is a binary code (clearly of length $n = n_1 n_2$). Compute the dimension and minimum distance of C.

Section C

Question 7. Prove that if G is a connected planar graph with n vertices, m edges, and girth at least g then

$$e \leq (n-2)\left(\frac{g}{g-2}\right).$$

[Hint: Use Euler's formula.] Prove that neither K_5 nor the Petersen graph is planar.

Question 8.

a) Suppose that x, y are distinct non-adjacent vertices in a graph G satisfying $d(x) + d(y) \geq n(G)$. Prove that $G + xy$ is Hamiltonian[1] if and only if G is.

b) A *Hamilton closure* of a graph G (having n vertices) is any graph obtained from G by iteratively joining non-adjacent vertices whose degrees sum to at least n until no such pair exists. Clearly (from the previous part) if C is a Hamilton closure of G then C is Hamiltonian if and only if G is. Prove that every graph G has a unique Hamilton closure.

Question 9. Recall that an *orientation* of a (simple) graph G is an assignment of a direction to each of its edges.

a) Prove that for some orientation of G every directed path has length at most $\chi(G) - 1$.

b) Prove that if D is any orientation of G then some directed path in G has length at least $\chi(G)-1$. [Hint: it might be helpful to consider a maximal subgraph of D having no directed cycles and finding a proper coloring of G based on it.]

[1] i.e., has a Hamilton cycle