Do six of the nine questions. Of these at least one should be from each section. If you work more than the required number of problems, make sure that you clearly mark which problems you want to have counted. Standard results may be used without proof provided they are clearly stated, though in no case should you interpret a problem in such a way that it becomes trivial. If you have doubts about the wording of a problem or about what results may be assumed without proof, please ask for clarification.

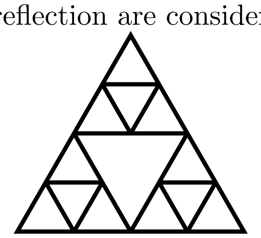
We write $[n]$ for the set $\{1, 2, \ldots, n\}$. If $\mathbb{F}$ is a finite alphabet and $\mathcal{C} \subseteq \mathbb{F}^n$ is a code we say that $\mathcal{C}$ is an (n, M, d) code if $|\mathcal{C}| = M$ and $\mathcal{C}$ has minimum distance d.

Section A

Question 1.

a) Prove the Orbit-Counting Lemma, concerning the number of orbits in the action of a group G on a set X.

b) How many ways are there to color the regions of the following diagram with colors red, green, and blue, if colorings that differ by a rotation or a reflection are considered equivalent?



Question 2.

a) Solve the linear recurrence

$$\begin{cases} a_n = 3a_{n-1} - 4a_{n-3} & n \geq 3 \\ a_0 = 2 \quad a_1 = 3 \quad a_2 = 13 \end{cases}$$

b) For which $\lambda \in \mathbb{R}$ is it possible to find a solution $(a_n)_{n \geq 1}$ to

$$a_n = -a_{n-1} + \lambda^n$$

that is unbounded as $n \to \infty$?

Question 3. The Catalan numbers, $(C_i)_{i=1}^{\infty}$, satisfy the recurrence

$$C_1 = 1, \qquad C_n = \sum_{i=1}^{n-1} C_i C_{n-i}.$$

Let $F(x) = \sum_{i=1}^{\infty} C_i x^i$ be the ordinary power series generating function of the sequence of Catalan numbers. Prove that $F(x) = x + (F(x))^2$, and hence (or otherwise) derive an explicit expressions for $F(x)$ and C_n.

Section B

Question 4. Let $V_q^n(r)$ be the volume of a Hamming ball of radius r in $\mathbb{F}_q^n$, so that

$$V_q^n(r) = \begin{cases} \sum_{i=0}^{r} \binom{n}{i}(q-1)^i & r \le n \\ q^n & r \ge n \end{cases}$$

Also define

$$M_q^n(d) = \max\{|C| : C \text{ is an } (n, M, d) \text{ code over } \mathbb{F}_q\}.$$

a) Prove that $M_q^n(d) \ge \dfrac{q^n}{V_q^n(d-1)}$.

b) Prove that $M_q^n(d) \le \dfrac{q^n}{V_q^n(\lfloor (d-1)/2 \rfloor)}$

Question 5. Suppose that $d \in \mathbb{N}$ is odd. Prove that a binary (n, M, d) code exists if and only if a binary $(n+1, M, d+1)$ code exists.

Question 6.

a) Prove Dilworth's theorem relating the sizes of antichains and chain decompositions in posets.

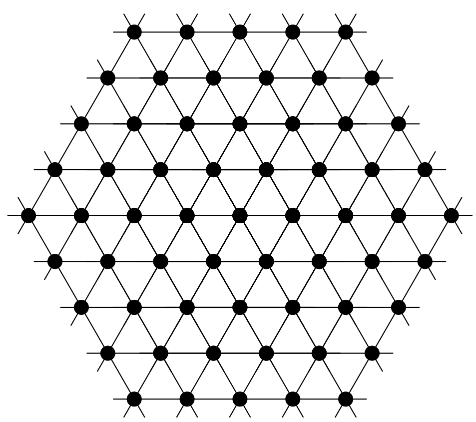
b) A family of sets $\mathcal{A}$ is called *union-free* if there do not exist distinct $A, B, C \in \mathcal{A}$ such that $A \cup B = C$. Prove that any family of sets $\mathcal{A}$ such that $|\mathcal{A}| = m$ contains a sub-family containing at least $\sqrt{m}$ sets that is union-free.

Section C

Question 7.

a) Recall that a graph G is *k-degenerate* if every subgraph H of G has $\delta(H) \le k$. Prove the Szekeres-Wilf theorem, that if G is a k-degenerate graph then $\chi(G) \le k+1$.

b) A portion of the infinite triangular lattice graph T is illustrated below. Prove that every finite induced subgraph of T is 4-colorable.



Question 8. The Ramsey number $R_k(3,3,\ldots,3) = R(\underbrace{3,3,\ldots,3}_{k \text{ 3s}})$ is the smallest n such that any coloring of the edges of K_n with k-colors contains a monochromatic triangle in one of the colors. Prove that

$$R_k(3,3,\ldots,3) \le \lfloor ek! \rfloor + 1.$$

[Hint: Here e is the base of natural logarithms; note that $\lfloor ek! \rfloor = 1 + k\lfloor e(k-1)! \rfloor$.]

Question 9. Prove that there exists a tournament on n vertices (that is, a directed graph on n vertices obtained by assigning an orientation to each edge of K_n) that has at least $n!/2^{n-1}$ spanning directed paths.