Do six of the nine questions. Of these at least one should be from each section. If you work more than the required number of problems, make sure that you clearly mark which problems you want to have counted. If you have doubts about the wording of a problem or about what results may be assumed without proof, please ask for clarification. In no case should you interpret a problem in such a way that it becomes trivial. We write $[n]$ for the set $\{1, 2, \dots, n\}$.

Section A

Question 1.

a) Solve the recurrence

$$\begin{cases} a_n = 6a_{n-1} - 9a_{n-2}, & n \geq 2 \\ a_0 = 1, \quad a_1 = 9 \end{cases}$$

b) Prove that if $f, g : \mathbb{N} \to \mathbb{R}$ are each individually the solution of some homogeneous linear recurrence with constant coefficients then so is their sum.

Question 2.

a) State and prove the Orbit Counting Lemma (sometimes called Burnside's Lemma), relating the number of orbits in a group action $G \rightarrowtail X$ to the the number of fixed points of the various group elements $g \in G$.

b) How many distinguishable 10 bead necklaces can be made using k colors of beads?

Question 3. This question concerns the Bell numbers $b(n)$, where $b(n)$ is number of (set) partitions of $\{1, 2, \dots, n\}$. Thus $b(0) = 1$, $b(1) = 1$, $b(2) = 2$, $b(3) = 5$, etc.

a) Prove that for $n \geq 0$ we have

$$b(n+1) = \sum_k \binom{n}{n-k} b(k).$$

b) Deduce that the exponential generating function, $B(x) = \sum_{n \geq 0} \frac{b(n)}{n!} x^n$, of $(b(n))_{n \geq 0}$ satisfies

$$\frac{d}{dx} B(x) = e^x B(x).$$

c) Use the previous part to prove that $B(x) = \exp(e^x - 1)$.

Section B

Question 4.

a) Prove Sperner's Lemma concerning the maximum size of an antichain in $\mathcal{P}(n)$.

b) Suppose that $\mathcal{A} \subseteq \mathcal{P}(n)$ is *convex*; that is to say that if $A \subseteq B \subseteq C$ with $A, C \in \mathcal{A}$ then necessarily $B \in \mathcal{A}$. Prove that

$$\sum_{A \in \mathcal{A}} (-1)^{|A|} \leq \binom{n}{\lfloor n/2 \rfloor}.$$

Question 5.

a) State (but you need not prove) Dilworth's theorem concerning chain decompositions of posets.

b) Let G be a graph with n vertices. Let D be an *orientation* of G — i.e. a directed graph obtained by assigning directions to the edges of G. Prove that if D has no directed cycles then D contains a directed path with at least $\lceil n/\alpha(G) \rceil$ vertices.

Question 6. The *binary Hamming code* Ham(r) of order r is the linear code over $\mathbb{F}_2$ of length $n = 2^r - 1$ whose parity check matrix H_r has for its columns all the $2^r - 1$ non-zero vectors in $\mathbb{F}_2^r$. Thus for instance

$$H_3 = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

a) Prove that any two columns of H_r are linearly independent, and deduce that the minimum distance of the Hamming code of order r is at least 3.

b) Prove, by using a counting argument or otherwise, that every vector in $\mathbb{F}_2^n$ is within distance 2 of a unique element of Ham(r)

Section C

Question 7.

a) State Menger's theorem concerning the existence of internally disjoint paths in k-connected graphs.

b) Suppose that G is a graph, $U \subseteq V(G)$, and $x \in V(G) \setminus U$. An $x - U$ *fan* is a set of $|U|$ paths from x to U such that any two of them have exactly the vertex x in common. Prove that if G is k-connected and $|U| = k$ then for all x in $V(G) \setminus U$ there is an $x - U$ fan in G.

c) Prove that if G is a k-connected graph ($k \geq 2$) with at least $2k$ vertices then there is a cycle in G of length at least $2k$.

Question 8. A *tournament* is a directed graph T such that for each pair x, y of vertices exactly one of $x \to y$ and $y \to x$ belongs to $E(T)$ (and there are no loops in T).

a) Prove that if there exists a tournament on n vertices with out-degrees $d_1 \leq d_2 \leq \cdots \leq d_n$ then we have

$$\sum_{i=1}^{k} d_i \geq \binom{k}{2} \quad \text{for all } 1 \leq k \leq n\text{; with equality for } k = n. \qquad (\star)$$

b) Suppose that $A_1, A_2, \ldots, A_n$ are disjoint sets with $|A_1| \leq |A_2| \leq \cdots \leq |A_n|$. Prove that if for all $1 \leq k \leq n$ we have

$$\sum_{i=1}^{k} |A_i| \geq \binom{k}{2}$$

then the family of sets $\left\{B_{ij} = A_i \cup A_j \ : \ \{i,j\} \in \binom{[n]}{2}\right\}$ has a system of distinct representatives.

c) Prove that condition $(\star)$ is not only necessary, but sufficient. I.e., if $d_1 \leq d_2 \leq \cdots \leq d_n$ is a sequence of non-negative integers satisfying $(\star)$ then there exists a tournament on n vertices with out degrees $d_1, d_2, \ldots, d_n$. [Hint: use the previous part.]

Question 9. Prove that if G is bipartite, then $\chi'(G) = \Delta(G)$. [Hint: Prove it first for a regular bipartite graph.]