Rules

1. There are ten questions divided into three sections (A,B,C). Each question is worth 20 points.
2. Do *six* of the questions. Of these *at least one* should be from each section. Do not turn in more than six problems. If these criteria are not followed, the first six problems that meet the criteria will be graded.
3. If you have doubts about the wording of a problem or about what results may be assumed without proof, please ask for clarification. In no case should you interpret a problem in such a way that it becomes trivial.

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For an indeterminate q, we define the "*q-integer*" $[n]_q$ as $[n]_q := 1 + q + \cdots + q^{n-1}$ if n is a positive integer and $[0]_q := 0$. The "*q-factorial*" is defined as the product of consecutive q-integers as $[n]_q! := [1]_q[2]_q \cdots [n]_q$ if n is a positive integer, and $[0]_q! := 1$. For a positive integer s and an integer t, the "*q-binomial coefficient*" is defined by

$$\binom{s}{t}_q := \frac{[s]_q!}{[t]_q![s-t]_q!}$$

if $s \geq t \geq 0$, and $\binom{s}{t}_q := 0$ if $s < t$ or $t < 0$.

Section A

Question 1. a) Prove the following identity for any positive integers m, n and any nonnegative integer r:

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}.$$

Note: Here $\binom{s}{t} = 0$ if $s < t$ or $t < 0$ by convention.

b) Prove or disprove the following q-analog of the identity in part (a) (for positive integers m, n and nonnegative integer r):

$$\binom{m+n}{r}_q = \sum_{k=0}^{r} \binom{m}{k}_q \binom{n}{r-k}_q q^{k(n-r+k)}.$$

Question 2. a) Prove the following *Cycle Lemma*:
Let r be a non-negative integer. For any binary sequence $\mathbf{s} = p_1p_2\cdots p_{m+n}$ of m 1's and n 0's, with $m \geq rn$, there exist exactly $m - rn$ *cyclic permutations* of $\mathbf{s}$ (a cyclic permutation of $\mathbf{s}$ is a sequence of the form $p_ip_{i+1}\cdots p_{m+n}p_1\cdots p_{i-1}$, for $1 \leq i \leq m+n$), that have the property that for all $j = 1, 2, \ldots, m+n$ the first j letters of this permutation contain more 1's than r times the number of 0's.

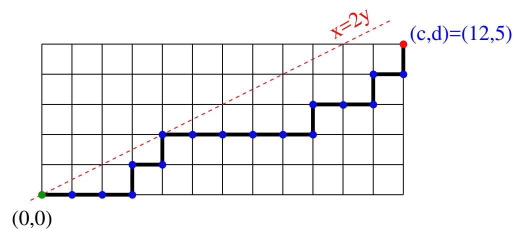

Figure 1: A Lattice path from $(0,0)$ to $(c,d) = (12,5)$ using unit up and unit right steps which lie weakly below the line $x = 2y$.

b) Use Cycle Lemma above to prove that for any nonnegative integers r, c, d with $c \geq rd$, the number of all lattice paths from the origin $(0,0)$ to the lattice point (c,d) using only unit up and unit right steps which lie weakly below the line $x = ry$ is given by

$$\frac{c - rd + 1}{c + d + 1}\binom{c + d + 1}{d}.$$

See Figure 1 for an example of the lattice path, for $c = 12$, $d = 5$, and $r = 2$.

Question 3. Let $S(n,k)$ be the *Stirling number of the second kind* for some positive integers n and k. Prove the following identity:

$$S(n,k) = \sum_{i=0}^{k} \frac{(-1)^i}{i!(k-i)!}(k-i)^n.$$

Section B

Question 4. a) Prove that a code with minimum distance d is e-error-correcting if and only if $d \geq 2e + 1$.

b) Show that a code with minimum distance d is capable of correcting any pattern of v errors and u erasures provided that $d \geq 2v + u + 1$.

Question 5. a) Suppose $d \geq 2e+1$. Prove that any q-ary code of length n and minimum distance at least d has at most

$$\frac{q^n}{\sum_{i=0}^{e}\binom{n}{i}(q-i)^i}$$

codewords.

b) Prove that a q-ary code of length n and minimum distance d has at least q^{n-d+1} codewords.

Question 6. Prove that for any sequence $(a_1, a_2, \ldots, a_{2022})$ of 2022 distinct real numbers, we can always find an increasing subsequence of length 43 or a decreasing subsequence of length 47.

Question 7. Let $a_1, a_2, \ldots, a_n$ be given real numbers of absolute value at least one. For any open unit interval I, there are at most $\binom{n}{\lfloor n/2 \rfloor}$ vectors $\mathbf{u} = (u_1, u_2, \ldots, u_n)$, where $u_i \in \{0,1\}^n$ for $1 \leq i \leq n$, such that $\sum_{i=1}^{n} u_i a_i \in I$.

Section C

All graphs are finite, undirected, and simple (i.e., there are no loops or multiple edges).

Question 8. Let T be a tree on n vertices. For each set of vertices $A \subseteq V(T)$, we denote $C(A)$ the number of connected components in the subgraph of T induced by A. Let k be a positive integer with $k \leq |V(T)|$. Compute the following sum:

$$\sum_{A \subseteq V(T), |A|=k} C(A).$$

Note: *Answering in the form of a complicated sum will NOT get credit.*

Question 9. Let G be a connected graph with chromatic number at least $k+1$, for some positive integer k. Prove that one can remove $\frac{k(k-1)}{2}$ edges from G without disconnecting it.

Question 10. Consider a bipartite graph G with (nonempty) bipartite sets A and B.

a) Prove that if there exists a positive integer k such that for every set $X \subseteq A$, we have $|N_G(X)| \geq k|X|$, then we can find $k|A|$ edges such that every vertex in A is incident to k of them and every vertex in B to at most one of them.

Hint: When $k = 1$, we get Hall's Theorem.

b) If there exists a positive integer d such that for every $X \subseteq A$, we have $|N_G(X)| \geq |X| - d$, then we can find $|A| - d$ pairwise non-adjacent edges.

Hint: When $d = 0$, we get Hall's Theorem.