Do six of the nine questions. Of these at least one should be from each section. If you work more than the required number of problems, make sure that you clearly mark which problems you want to have counted. If you have doubts about the wording of a problem or about what results may be assumed without proof, please ask for clarification. In no case should you interpret a problem in such a way that it becomes trivial. We write $[n]$ for the set $\{1, 2, \ldots, n\}$.

Section A

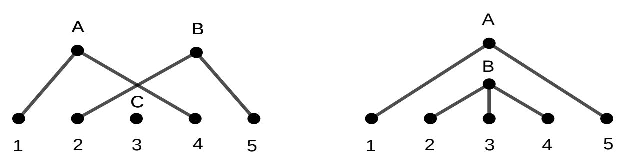
Question 1. By counting a set in two ways, prove that

$$\sum_{A\subseteq[n]} \sum_{B\subseteq[n]} |A \cap B| = n4^{n-1}.$$

Question 2. Evaluate the sum below using generating functions, and then give an inclusion-exclusion proof of the resulting identity.

$$\sum_{k=0}^{m} (-1)^k \binom{n}{k} \binom{n-k}{m-k}.$$

Question 3. A partition of $[n]$ is *noncrossing* if there are no a, b, c, d with $a < b < c < d$ such that a and c are in one block and b and d are in another. Thus, $(14|25|3)$ is crossing and $(15|234)$ is noncrossing (see illustration below). Establish a bijection from the set of noncrossing partitions of $[n]$ to the set of ballot lists of length $2n$. From the bijection, argue that the number of noncrossing partitions of $[n]$ with k blocks equals the number of ballot lists of length $2n$ with k runs of 1s. (*Recall that a* ballot list *of length* $2n$ *is a list of* n *1s and* n *0s such that each initial segment has at least as many 1s as 0s.*)



Section B

Question 4.

(a) What does it mean for a linear code to be maximum distance separable (MDS)?

(b) Prove that every Reed-Solomon code is MDS.

Question 5.

(a) State (but you need not prove) Erdös–Ko–Rado's theorem.

(b) Construct a family of r-subsets of $[n]$ achieving the Erdös–Ko–Rado bound on cardinality.

Question 6.

(a) State (but you need not prove) Dilworth's theorem concerning chain decompositions of posets.

(b) Use Dilworth's theorem to deduce the following König's theorem:

König's theorem. *In any bipartite graph, the number of edges in a maximum matching is equal to the number of vertices in a minimum vertex cover.*

Section C

All graphs are finite, undirected, and simple (i.e., there are no loops or multiple edges).

Question 7. Assume that $n \geq 2$ is a positive integer. Let G be a simple graph on n vertices. Denote by $\overline{G}$ the complement graph of G. Prove the following double-inequality for the chromatic numbers of G and $\overline{G}$:

$$2\sqrt{n} \leq \chi(G) + \chi(\overline{G}) \leq n + 1.$$

Hint: For all real numbers a, b, we always have $a^2 + b^2 \geq 2ab$.

Question 8. Assume that G is a 2-connected graph. Assume that C_1 and C_2 are two cycles of maximum length in G. (In particular, C_1 and C_2 have the same length.) Prove that C_1 and C_2 have at least two vertices in common.

Question 9.

(a) Let T_1 and T_2 be two spanning trees of a connected graph G. Prove that T_1 can be transformed into T_2 through a sequence of intermediate trees, each arising from the previous one by removing an edge and adding another.

(b) Suppose that G is 2-connected. Then to obtain T_2 from T_1, it suffices to apply a sequence of the following transformation: *we remove the edge adjacent to some leaf u of the tree and connect u to the rest by some other edge of G.*