Do six of the nine questions. Of these at least one should be from each section. If you work more than the required number of problems, make sure that you clearly mark which problems you want to have counted. If you have doubts about the wording of a problem or about what results may be assumed without proof, please ask for clarification. In no case should you interpret a problem in such a way that it becomes trivial. We write $[n]$ for the set $\{1, 2, \dots, n\}$.

Section A

Question 1. The sequence $(a_0, a_1, a_2, \dots)$ is defined by

$$\begin{cases} a_{n+1} = \sum_{i=0}^{n} a_i a_{n-i}, & n \geq 0 \\ a_0 = 1 \end{cases}$$

a) Find the generating function $A(x) = \sum_{n=0}^{\infty} a_n x^n$.
b) Use part (a) to find an explicit formula for a_n. Simplify your answer the best you can. Answering by a complicated sum or product will not give you full credit.
c) Prove that a_n counts the number of *triangulations* of a convex $(n+2)$-gon (i.e. the ways to divide a convex $(n+2)$-gon into triangles by non-intersecting diagonals).

Question 2.
a) State and prove the *Orbit Counting Lemma* (sometimes called *Burnside's Lemma*), relating the number of orbits in a group action $G \rightarrowtail X$ to the number of fixed points of the various group elements $g \in G$.
Note: You can use (without proving) fundamental algebraic results such as the *Orbit–Stabilizer Theorem.*
b) The 64 squares of an 8×8 chess board are colored using n different colors (each square has one color, but a color may be used in many squares). How many different colorings are there if two colorings that can be obtained from each other by rotation and/or reflection are identical?

Question 3.
a) A *derangement* is a permutation that has no fixed points. Denote by D_n the number of derangements of the set $[n]$. Use Inclusion–Exclusion Principle to show that

$$D_n = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}.$$

b) Let w_n be the number of *involutions* on $[n]$ (i.e. permutations that equal their inverse). Prove that

$$w_n = \sum_{k \geq 0} \binom{n}{2k} \frac{(2k)!}{(k! 2^k)}.$$

Hint: Investigate the cycle structure of an involution.

Section B

Question 4.

a) Prove the following *Lubell–Yamamoto–Meshalkin Inequality* (a.k.a. "*LYM Inequality*"):
"Let U be an n-element set and $\mathcal{A}$ be a family of subsets of U such that no set in $\mathcal{A}$ is a subset of another set in $\mathcal{A}$. Let a_k be the number of k-sets in $\mathcal{A}$. Then

$$\sum_{k=0}^{n} \frac{a_k}{\binom{n}{k}} \leq 1."$$

Hint: Count permutations of U in two different ways.

b) Use the LYM inequality to prove Sperner's Lemma concerning the maximum size of an antichain in $\mathcal{P}(n)$.

Question 5.

a) State (but you need not prove) Dilworth's theorem concerning chain decompositions of posets.

b) Let m, n be positive integers. Prove that every sequence of $mn + 1$ distinct integers contain an increasing subsequence of length at least $m + 1$ or a decreasing subsequence of length at least $n + 1$.

Question 6.

a) Give a definition of the capacity of a (noisy) channel C.

b) Consider two discrete memoryless channels C_1 and C_2, each taking input from alphabet A and giving output from alphabet B. Channel i is specified by its channel probabilities $p_i(y \mid x)$ for $x \in A$, $y \in B$; the probability that y is received given that x is transmitted. The *product* of C_1 and C_2 is the channel with input alphabet A^2 and output alphabet B^2 with channel probabilities

$$p\big((y_1, y_2) \mid (x_1, x_2)\big) = p_1(y_1 \mid x_1)\, p_2(y_2 \mid x_2).$$

Prove that the capacity of the product is the sum of the capacities of the channels.

Section C

Question 7.

a) Let v be a vertex of degree 5 in a planar graph G, and let x, y be two nonadjacent neighbors of v. Let G' be the graph obtained by contracting the edges vx and vy. Prove that if G' is 5-colorable, then G is also 5-colorable.

b) Use part a) to prove the Five Color Theorem.

Question 8.

a) State and prove Ore's theorem giving a sufficient condition for a graph to be Hamiltonian.

b) Let G be a graph that is not a forest and has girth at least 5 (recall that the girth is the length of the shortest cycle). Use Ore's theorem to show that $\overline{G}$ is Hamiltonian.
Hint: You may want to consider first the case where the girth of G is at least 6.

Question 9.

a) State Hall's theorem concerning the existence of a perfect matching in a bipartite graph.

b) A $k \times n$ *Latin rectangle* is a $k \times n$ matrix with entries from $[n]$ such that each $i \in [n]$ appears at most once in each row and column. Prove that if $k < n$ then any $k \times n$ Latin rectangle can be extended to a $(k + 1) \times n$ Latin rectangle.