Do six of the nine questions. Of these at least one should be from each section. If you work more than the required number of problems, make sure that you clearly mark which problems you want to have counted. Standard results may be used without proof provided they are clearly stated, though in no case should you interpret a problem in such a way that it becomes trivial. If you have doubts about the wording of a problem or about what results may be assumed without proof, please ask for clarification.

We write $[n]$ for the set $\{1, 2, 3, \ldots, n\}$, and let $\langle h \rangle$ denote the sequence $h_0, h_1, h_2, \ldots$.

Section A

1. State and prove the Inclusion-Exclusion principle. Then use it to prove that $S(n, k)$, the number of partitions of an n-element set into k parts, is

$$S(n, k) = \frac{1}{k!} \sum_{i=0}^{k} \binom{k}{i} (k - i)^n.$$

2. Let a_n be the number of n-tuples in $[4]^n$ that have at least one 1 and have no 2 appearing before the first 1. (note that the sequence $\langle a \rangle$ begins with $0, 1, 6, \ldots$.)

   (a) Obtain and solve a recurrence for $\langle a \rangle$.

   (b) Give a direct counting argument (without using summations) to prove the resulting formula.

3. The parts of this question are unrelated.

   (a) Evaluate the sum below by recognizing it as a convolution of two sequences,

$$\sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k}.$$

   (b) Let c_n denote the number of ways that c children can be arranged in teams, with a captain for each team chosen from the team members. Prove that the exponential generating function for $\langle c \rangle$ is e^{xe^x}.

Section B

4. (a) Let C be a linear $[n, k, d]$ code over $\mathbb{F}_q$ and let T be a $q^k \times n$ array whose rows are the codewords of C. Show that each element of $\mathbb{F}_q$ appears in every nonzero column of T exactly q^{k-1} times.

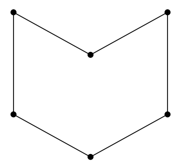
(b) Show that every linear $[n, k, d]$ code over $\mathbb{F}_q$ satisfies

$$d \leq \frac{n(q-1)q^{k-1}}{q^k - 1}.$$

(Hint: Consider the average Hamming weight of the nonzero codewords).

5. a) Define the *dimension* of a poset P.

   b) Determine, with proof, the dimension of the poset whose Hasse diagram is shown below.



6. Let G be the incidence graph of the Fano plane; G is bipartite and has 14 vertices. Using the properties of the Fano plane (without drawing G), determine the minimum size of a dominating set in G. (A *dominating set* in a graph G is a subset S of $V(G)$ such that every vertex $v \in V(G) - S$ has a neighbor in S.)

Section C

7. Show that if G is a bipartite graph, then its complement $\overline{G}$ is perfect.

8. Use the vertex version of Menger's theorem to prove the König–Egerváry theorem.

9. Show that there is a constant K such that if G is a 3-regular connected plane graph whose faces are all hexagons or pentagons, then exactly K of the faces of G are pentagons. Determine the value of K.