Do six of the nine questions. You should do at least one question from each section. If you have doubts about the wording of a problem or about what results may be assumed without proof, please ask for clarification. In no case should you interpret a problem in such a way that it becomes trivial. We write $[n]$ for the set $\{1, 2, \ldots, n\}$.

Section A

Question 1. Recall that if $\pi = \pi_1\pi_2 \ldots \pi_n$ is a permutation of $[n] = \{1, 2, \ldots, n\}$ then we say that the pair of indices (i, j) is an *inversion of* π if $i < j$ but $\pi_i > \pi_j$. For $n \geq 1$ let I_n be the total number of inversions in all permutations of $[n]$. For instance, $I_1 = 0$, $I_2 = 1$, and $I_3 = 9$.

a) Prove that for all positive integers n we have

$$I_{n+1} = (n+1)I_n + n!\binom{n+1}{2}.$$

b) Use the above recurrence to deduce that for all $n \geq 1$

$$I_n = \frac{n!}{2}\binom{n}{2}.$$

Question 2. Let $d_{n,k}$ be the number of *ordered* sequences of k die rolls (i.e., sequences of integers from 1 to 6) that add up to n. Let d_n be the number of ordered sequences of die rolls of any length adding up to n. Thus $d_{5,2} = 4$ since 5 can be written as $1+4$, $2+3$, $3+2$, or $4+1$. Similarly we have $d_4 = 8$ because a total of 4 can be rolled in 8 ways:

$$4, \quad 3+1, \quad 1+3, \quad 2+2, \quad 2+1+1, \quad 1+2+1, \quad 1+1+2, \text{ and } \quad 1+1+1+1.$$

a) Determine an explicit form for the generating function $D_k(x) = \sum_{n\geq 0} d_{n,k}x^n$.

b) Determine an explicit form for the generating function $D(x) = \sum_{n\geq 0} d_n x^n$. Express your answer as a rational function.

Question 3.

a) State and prove the Orbit Counting Lemma (sometimes called Burnside's Lemma), relating the number of orbits in a group action $G \rightarrowtail X$ to the the number of fixed points of the various group elements $g \in G$.

b) How many distinguishable 10 bead necklaces can be made using k colors of beads?

Section B

Question 4. Prove that if $1 \leq d \leq n-1$ and q is a prime power then there is a code over the finite field $\mathbb{F}_q$ of length n and minimum distance at least d having at least

$$q^n \Big/ \left(\sum_{i=0}^{d-1}\binom{n}{i}(q-1)^i\right)$$

codewords. [Note that we do not require that the code be linear.]

Question 5. Let q be a prime power. Fix $1 \leq k < n$, and suppose that $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{F}_q$ are distinct. (Here $\mathbb{F}_q$ is the field with q elements so in particular $n \leq q$.) Let $H \in M_{(n-k)\times n}(\mathbb{F}_q)$ be defined by

$$H = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \alpha_1^2 & \alpha_2^2 & \cdots & \alpha_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1^{n-k-1} & \alpha_2^{n-k-1} & \cdots & \alpha_n^{n-k-1} \end{pmatrix}.$$

a) Prove that the rows of H are linearly independent, and deduce that H is the parity check matrix for a linear code C whose length and dimension you should determine.

b) Further show every set of $n-k$ columns of H are linearly independent, and deduce that C has minimum distance at least $n-k+1$

Question 6. Recall that we call a family of subsets $\mathcal{F} \subseteq \mathcal{P}(n)$ a *k-family* if no $k+1$ sets $A_1, A_2, \ldots, A_{k+1} \in \mathcal{F}$ satisfy

$$A_1 \subsetneq A_2 \subsetneq \cdots \subsetneq A_{k+1}.$$

Let $\mathcal{F}$ be a k-family in $\mathcal{P}(n)$.

a) Prove that if σ is a circular permutation of $[n]$ then at most kn sets from $\mathcal{F}$ are intervals with respect to σ.

b) Prove that if $\mathcal{F}$ contains a_i sets of size i then

$$\sum_{i=0}^{n} \frac{a_i}{\binom{n}{i}} \leq k.$$

[Hint: you might want to first prove the case where $\emptyset, [n] \notin \mathcal{F}$.]

Section C

Question 7.

a) State and prove Hall's Theorem concerning the existence of matchings saturating X in a bipartite graph with bipartition X, Y.

b) Given a sequence $\mathcal{A} = (A_i)_1^m$ of subsets of $[n]$, we say that $\mathcal{A}$ has a *system of distinct representatives* if there exist $x_1, x_2, \ldots, x_m$, all distinct, such that $x_i \in A_i$ for all $i \in [m]$. Prove that $\mathcal{A}$ has a system of distinct representatives if and only if for all $S \subseteq [m]$

$$\left|\bigcup_{i\in S} A_i\right| \geq |S|.$$

Question 8. Let G be a 2-connected graph. Prove that for all $v \in V(G)$ there exists $u \in V(G)$ such that u is adjacent to v and also $G \setminus \{u, v\}$ is connected.

Question 9. Let G be a graph that does not contain two disjoint odd cycles. Prove that $\chi(G) \leq 5$.