Math 871-872 Qualifying Exam
January 2020

Do three questions from Section A and three questions from Section B. You may work on as many questions as you wish, but you must indicate which ones you want graded. When in doubt about the wording of a problem or what results may be assumed without proof, ask for clarification. Do not interpret a problem in such a way that it becomes trivial.

Section A: Do THREE problems from this section.

A1. Suppose Y is Hausdorff, and $f : X \to Y$ and $g : X \to Y$ are continuous functions. Prove that the set $A = \{x \in X : f(x) = g(x)\}$ is closed in X.

A2. (a) State the definition of what it means for a topological space X to be regular.

(b) Prove that if X and Y are regular, then the space $X \times Y$ with the product topology is regular.

A3. Suppose that X is compact and nonempty, and Y is Hausdorff and connected. Prove that any continuous, open map $f : X \to Y$ is onto.

A4. Let $\mathcal{B} = \{[a, b) : a, b \in \mathbb{R}\}$; let $\mathbb{R}_\ell$ denote $\mathbb{R}$ equipped with the topology generated by the basis $\mathcal{B}$ (you do not need to prove $\mathcal{B}$ is a basis). Decide (with proof) if the following sets are compact in $\mathbb{R}_\ell$:

(a) $A = \{1/n : n \in \mathbb{N}\} \cup \{0\}$.

(b) $B = \{-1/n : n \in \mathbb{N}\} \cup \{0\}$.

Section B: Do THREE problems from this section.

B1. Prove that every contractible space is path-connected. [Caution: X being contractible does *not* imply that X strong deformation retracts to a point.]

B2. Suppose Y is Hausdorff, path-connected, locally path-connected, and semilocally simply-connected, and that its universal cover $\widetilde{Y}$ is compact. Prove that $\pi_1(Y)$ is finite.

B3. The free group $F(a, b)$ on two generators is the fundamental group of the space $Z = S^1 \vee S^1$, using the wedge point as basepoint, with a and b corresponding to each of the loops S^1. Construct the covering space of Z corresponding to the subgroup $H = \langle a^2b, aba, ab^2, b^2a\rangle$, determine the index of H in $F(a, b)$, and decide if H is a normal subgroup of $F(a, b)$.

B4. Consider the simple closed curve A contained in the torus T, as shown below. Sometimes A is a called a $(1, 1)$-*curve*. Use the long exact sequence of relative homology groups to compute $H_n(T, A)$ for all n. (You may state the homology groups $\widetilde{H}_n(T)$ and $\widetilde{H}_n(S^1)$ without proof.)