Math 871-872 Qualifying Exam
June 2020

Do three questions from Section A and three questions from Section B. You may work on as many questions as you wish, but you must indicate which ones you want graded.

Standard results may be quoted provided that they are stated clearly and do not trivialize the problem. When in doubt about the wording of a problem or what results may be assumed without proof, ask for clarification. Do **not** interpret a problem in such a way that it becomes trivial.

Section A: Do THREE problems from this section.

A1. Let $f, g : X \to Y$ be two continuous maps, where X is any topological space and Y is a Hausdorff topological space. Prove that $W = \{x \in X : f(x) = g(x)\}$ is a closed subspace of X. Deduce that the set of fixed points of any continuous map of a Hausdorff topological space to itself is closed.

A2. Let $(K_i)_{i=1}^{\infty}$ be compact subsets of a Hausdorff topological space X.

a) Prove that if U is open in X and $\bigcap_{i=1}^{\infty} K_i \subseteq U$ then there exists $n \in \mathbb{N}$ such that $\bigcap_{i=1}^{n} K_i \subseteq U$.

b) Deduce that if X is a metric space and $\bigcap_{i=1}^{\infty} K_i = \{x\}$ then

$$\operatorname{diam}\Big(\bigcap_{i=1}^{n} K_i\Big) \to 0 \qquad \text{as } n \to \infty.$$

A3. We say that a function $f : X \to Y$ between topological spaces is *locally constant* if for all $x \in X$ there exists an open set U containing x such that $f|_U$ is constant.

a) Define what it means for a topological space to be connected.

b) Prove that all locally constant functions are continuous.

c) Prove that X is connected iff all locally constant functions $f : X \to Y$ are constant.

A4. Recall that a topological space X is *separable* if it has a countable dense subset. Also a topological space X is *first countable* if it has a countable neighborhood basis at each point $x \in X$. A *countable neighborhood basis at* x is a collection $\{B_i\}_{i\in\mathbb{N}}$ of open sets containing x such that for any open set U containing x there exists $i \in \mathbb{N}$ such that $B_i \subseteq U$.

a) Prove that the continuous image of a separable space is separable.

b) Suppose that $f : X \twoheadrightarrow Y$ is a continuous, open, surjective map. Prove that if X is first countable then so is Y.

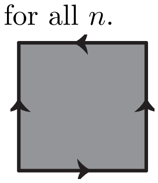
Section B: Do THREE problems from this section.

B1. Let K a graph with two vertices v_0, v_1 and three edges: e_0 from v_0 to v_1, e_1 from v_1 to v_0, and e_2 from v_0 to v_0. Construct a CW complex X by attaching two 2-cells to K, one along the loop $e_0 \cdot e_1 \cdot e_2$ and one along $e_0 \cdot e_1 \cdot \overline{e_2}$. Compute $\pi_1(X)$ and show it is isomorphic to a familiar group.

B2. State and prove the path lifting property.

B3. Fix an arbitrary integer $n > 1$ and denote $T^n = S^1 \times \cdots \times S^1$ (n copies). Prove that every continuous map $f : S^n \to T^n$ is nullhomotopic. (State any properties of S^n that you use.)

B4. Let X be the space obtained by identifying all four edges of the unit square as shown. That is, $X = I \times I/(0,a) \sim (a,0) \sim (1,a) \sim (1-a,1)$. Describe a Δ-complex structure for X, and compute the simplicial homology groups $H_n^{\Delta}(X)$ for all n.



(Your description of a Δ-complex structure need only contain enough detail to describe your homology calculations; a labeled picture is a good option.)