Ph.D. Qualifying Exam
Topology and Algebraic Topology
Math 970-971, January 18, 2002

Do a total of five (5) problems, three (3) from section A and two (2) from section B. All problems count equally. If you work on more than the required number of problems in each section, then you should clearly mark which problems in each section you wish to have graded. Otherwise, the first problems from each section will be graded.
If you have doubts about the wording of a problem, please ask for clarification. In no case should you interpret a problem in such a way that its solution becomes trivial.

Section A: General Topology

A1. Let X be a compact space, and let

$$A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq \cdots$$

be a descending chain of non-empty closed subsets of X. Show that their intersection

$$\bigcap_{n=1}^{\infty} A_n \qquad \text{is non-empty.}$$

A2. Let X be a topological space. A set $A \subseteq X$ is called *nowhere dense* if the closure $\bar{A}$ of A has empty interior, i.e., $\mathrm{int}(\bar{A}) = \emptyset$.

Show that if $U \subseteq X$ is open, then $A = \bar{U} \setminus U$ is nowhere dense.

A3. Let X be a topological space, and $A, B \subseteq X$ be connected subsets of X. Show that if $A \cap \bar{B} \neq \emptyset$, then $A \cup B$ is a connected subset of X.

A4. If (X, d) is a metric space, and $A, B \subseteq X$ are non-empty, then we define the *distance between A and B* by

$$d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$$

where $d : X \times X \to \mathbb{R}$ is the metric.

Show that if A and B are both compact and non-empty, then there are $a_0 \in A$ and $b_0 \in B$ so that $d(A, B) = d(a_0, b_0)$.

A5. Show that $\mathbb{R}$ and $\mathbb{R}^2$ (with their usual topologies) are not homeomorphic.

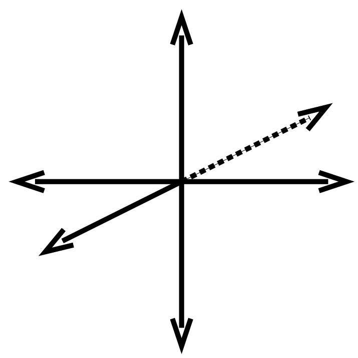
Section B: Algebraic Topology

B1. Show that if X is path-connected, locally path-connected space, with $\pi_1(X)$ finite, then every continuous map

$$f : X \to S^1$$

is homotopic to a constant map.

B2. Find the fundamental group of the space X consisting of $\mathbb{R}^3$ with the three coordinate axes removed; see figure.



B3. Let X be the 2-sphere, and $A \subseteq X$ the equatorial circle in X. Show that there is no retraction $r : X \to A$.

B4. Let $p : \widetilde{X} \to X$ be a covering map, with X and locally path-connected, and $x_0 \in X$. Show that for $y_0, y_1 \in p^{-1}(\{x_0\})$, there is a deck transformation of $\widetilde{X}$ taking y_0 to y_1 if and only if $p_*(\pi_1(\widetilde{X}, y_0)) = p_*(\pi_1(\widetilde{X}, y_1))$.