Do seven questions. Of these at least three should be from section A and at least three from section B. If you work more than the required number of problems, make sure that you clearly mark which problems you want to have counted. Standard results from the courses may be used without proof provided they are clearly stated. If you have doubts about the wording of a problem or about what results may be assumed without proof, please ask for clarification. In no case should you interpret a problem in such a way that it becomes trivial.

Section A.

Question 1. Suppose that d,d' are metrics on the set X with $d(x,y) \le d'(x,y)$ for every $x,y \in X$. Show that the metric topology (X,d) is *coarser* than the metric topology (X,d').

Question 2. A toplogical space $(X,\mathcal{T})$ is called *limit-point compact* if every infinite subset A of X has a limit point. Show that every closed subset of a limit-point compact space is limit-point compact.

Question 3. Let $S^1 \subseteq \mathbb{R}^2$ denote the unit sphere (with the subspace topology), and $\mathbb{R}$ the real line with the usual topology. Show that for every continuous map $f : S^1 \to \mathbb{R}$ there is an $x \in S^1$ with $f(x) = f(-x)$.

Question 4. Let $\mathcal{T},\mathcal{T}'$ be two topologies on X. Show that if $(X,\mathcal{T})$ is compact and Hausdorff, $\mathcal{T} \subseteq \mathcal{T}'$, and $\mathcal{T} \neq \mathcal{T}'$, then $(X,\mathcal{T}')$ is Hausdorff but *not* compact.

Question 5. Recall that a topological space X is *locally connected* if for every point $x \in X$ and every neighbourhood U of x there exists a connected neighbourhood V of x with $V \subseteq U$.
a. Prove that a topological space X is locally connected iff for every open set $U \subseteq X$ the components of U are open. Now let $p : X \to Y$ be a quotient map.
b. Prove that if C is a component of an open subset $U \subseteq Y$ then $p^{-1}(C)$ is a union of components of $p^{-1}(U)$.
c. Deduce that if X is locally connected then so is Y.

Section B.

Question 6. Find a cell structure for, and compute a presentation for, the fundamental group of the space X described below. X is a quotient space of the disjoint union of two copies $T_1 = T_2 = S^1 \times S^1$ of the 2-torus. Fix a basepoint x_0 in S^1. X is obtained by gluing T_1 and T_2 together, identifying (x_0, x) in T_1 with (x, x_0) in T_2, for each $x \in S^1$. For concreteness, use (x_0, x_0) as your basepoint for X.

Question 7. Let $p : \widetilde{X} \to X$ be a covering map, and $f : Y \to X$ be a continuous map. Define $\widetilde{Y} = \left\{(y, \widetilde{x}) \in Y \times \widetilde{X} \ : \ f(y) = p(\widetilde{x})\right\} \subseteq Y \times \widetilde{X}$, with the subspace topology inherited from $Y \times \widetilde{X}$, and define $q : \widetilde{Y} \to Y$ by $q(y, \widetilde{x}) = y$. Show that q is also a covering map.

Question 8. Find a Δ-complex structure on the space X obtained by identifying the boundaries of three copies of the unit disk together (using identity maps), and compute the simplicial homology groups of X.

Question 9. Use Mayer-Vietoris sequences to compute the singular homology groups of the 2-torus $T = S^1 \times S^1$. (You may use knowledge of the homology groups of the circle S^1 in your calculations.)