Do three (3) problems from each section. If you work on more than the required number of problems, make sure that you clearly mark which problems you want to have counted. Standard results from the courses may be used without proof provided they are clearly marked. If you have doubts about the wording of a problem or about what results may be assumed without proof, please ask for clarification. In no case should you interpret a problem in such a way that it becomes trivial.

Section A.

A1. A space $(X, \mathcal{T})$ is called *extremally disconnected* if the closure of every open set is open; that is, whenever $\mathcal{U} \in \mathcal{T}$ we have $\mathrm{cl}(\mathcal{U}) \in \mathcal{T}$. Show that in an extremally disconnected space, if $\mathcal{U}, \mathcal{V} \in \mathcal{T}$ and $\mathcal{U} \cap \mathcal{V} = \emptyset$, then $\mathrm{cl}(\mathcal{U}) \cap \mathrm{cl}(\mathcal{V}) = \emptyset$.
(Hint: first show that $\mathcal{U} \cap \mathrm{cl}(\mathcal{V}) = \emptyset$.)

A2. A space $(X, \mathcal{T})$ is called *locally path-connected* if, for every $x \in X$, every neighborhood of x contains a path-connected neighborhood of x. Show that a connected, locally path-connected space is path-connected.

A3. Suppose that X is compact, Y is Hausdorff, $f : X \to Y$ is a continuous surjective map, and $g : Y \to Z$ is a function. Show that if $g \circ f : X \to Z$ is continuous, then g is continuous.

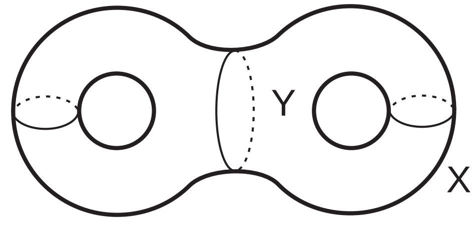
A4. A space $(X, \mathcal{T})$ is called *locally compact* if, for every $x \in X$ and $x \in \mathcal{U} \in \mathcal{T}$, there is a compact set $C \subseteq \mathcal{U}$ containing an open neighborhood of x. Show that the Cartesian product of two locally compact spaces is locally compact.

Section B.

B1. Let $X = T^2 = S^1 \times S^1$ and $X = \mathcal{U}_1 \cup \cdots \cup \mathcal{U}_n$ where each $\mathcal{U}_i$ is open and simply-connected. Show that there is an i for which $\mathcal{V}_i := \mathcal{U}_i \cap (\mathcal{U}_1 \cup \cdots \cup \mathcal{U}_{i-1})$ is either empty or not path-connected. (Hint: argue by contradiction.)

B2. Let $p : (\widetilde{X}, y_0) \to (X, x_0)$ be a covering space projection, $x_0 \in A \subseteq X$, and $\iota : A \to X$ the inclusion map. Show that $q = p|_{p^{-1}(A)} : (p^{-1}(A), y_0) \to (A, x_0)$ is also a covering space, and $\ker(\iota_*) \subseteq \mathrm{im}(q_*) \subseteq \pi_1(A, x_0)$.

B3. Let X be the surface of genus 2 shown below and let $Y \supseteq X$ be the region of $\mathbb{R}^3$ that it encloses. Show that there is no retraction $r : Y \to X$ (that is, no continuous map for which $r(x) = x$ for all $x \in X$).



B4. Find the Euler characteristic of $X = (\Delta^5)^{(2)}$, the 2-skeleton of the 5-simplex. Show that $H_1(X) = 0$, $H_2(X)$ is free abelian, and compute the rank of $H_2(X)$.