

INSTRUCTIONS: Do three of the problems from section A and three questions from section B. If you work more than the required number of problems, make sure that you clearly mark which problems you want to have graded. If you have doubts about the wording of a problem or about what results may be assumed without proof, please ask for clarification. In no case should you interpret a problem in such a way that it becomes trivial. Each problem is worth 20 points.

Section A:

- 1) Assume that (X, τ) is a topological space with the property that for every open set $G \subseteq X$, the closure of G , \overline{G} , is open. Such topological spaces are called *extremally disconnected*. Prove the following.
 - a) If $F \subseteq X$ is a closed set, then the interior of F , F° , is closed.
 - b) If $G \subseteq X$ is an open set, then G equipped with the subspace topology is extremally disconnected.
 - c) If X is Hausdorff and $f : \mathbb{R} \rightarrow X$ is continuous, prove that f is a constant function.

- 2) Suppose X and Y are topological spaces and $f : X \rightarrow Y$ is a continuous function.
 - a) Prove there exists a topological space Z , a quotient map $q : X \rightarrow Z$ and a one-to-one continuous map $h : Z \rightarrow Y$ such that $f = h \circ q$.
 - b) (This part is a continuation of part (a).) If Z_1 is a topological space, $q_1 : X \rightarrow Z_1$ is a quotient map, $h_1 : Z_1 \rightarrow Y$ is one-to-one and continuous, and $f = h_1 \circ q_1$, prove that Z and Z_1 are homeomorphic.

- 3) The parts of this question are unrelated.
 - a) Suppose $A \subseteq \mathbb{R}^n$ is non-empty and there exists $a_0 \in A$ such that for every $a \in A$ the line segment joining a_0 to a lies in A . Prove that A is a contractible space.
 - b) Suppose that X is a topological space and for $i = 0, 1$, f_i and g_i are paths in X such that the products, $f_i * g_i$ are defined and $f_0 * g_0$ and $f_1 * g_1$ are path homotopic. Prove that if g_0 and g_1 are path homotopic, then f_0 and f_1 are path homotopic. You may use standard theorems of path homotopy, but if you do, you must state and prove at least one of the theorems you use by explicitly constructing a homotopy.

- 4) Give a complete and careful proof of the following well-known theorem: *If X is a compact Hausdorff space, then X is normal.*

(Section B is on the reverse.)

Section B:

5) Let X be the space obtained from the torus $T = S^1 \times S^1$ by gluing in a disk D^2 along its boundary $\partial D^2 = S^1$ using the map $\alpha : S^1 \rightarrow T$ given by $z \mapsto (z, (1, 0))$ for $z \in S^1$.

- a) Find $\pi_1(X)$.
- b) Give a CW complex for X .

6) Suppose G is a finite group acting on a Hausdorff space X and assume the action is fixed point free (i.e., if $g \in G$ is not the identity element, then $g \cdot x \neq x$ for all $x \in X$).

- a) Prove that the action is necessarily a covering space action. (*Note:* Recall that this means every point $x \in X$ has an open neighborhood U such that $U \cap g \cdot U = \emptyset$ for all non-identity elements $g \in G$.)
- b) Assume X is simply connected and let X/G be the quotient space. Prove every map from X/G to S^1 is contractible.

7) Let $X = \mathbb{R}P^2 \vee \mathbb{R}P^2$, i.e., the result of gluing two copies of the real projective plane together along specified base points. *Note:* Recall that $\mathbb{R}P^2$ is the quotient of S^2 by the antipodal map.

- a) Find $\pi_1(X)$.
- b) Find the homology of X .

8) X is the space obtained by gluing a disk to a sphere as follows: Let $\alpha : S^1 \rightarrow S^2$ be the map that “wraps S^1 around the equator of S^2 three times” — more explicitly, α is the composition of the map $S^1 \rightarrow S^1$ sending z to z^3 (where S^1 is identified with complex numbers of norm one) and the canonical inclusion $S^1 \subseteq S^2$. We glue D^2 onto S^2 by attaching its boundary $\partial D^2 = S^1$ according to α :

$$X = (S^2 \amalg D^2) / \sim$$

where \sim is the equivalence relation generated by $\alpha(z) \sim z$ for all $z \in \partial D^2 = S^1$.

Compute the singular homology of X .