Math 871-872 Qualifying Exam

June 2022

Do three questions from Section A and three questions from Section B. You may work on as many questions as you wish, but you must indicate which ones you want graded.

Standard results may be used if the **statement of the result is clearly stated** and does not trivialize the problem. When in doubt about the wording of a problem or what results may be assumed without proof, ask for clarification. Do not interpret a problem in such a way that it becomes trivial.

Section A: Do THREE problems from this section.

A1. Let X be a compact topological space. Prove that if

$$C_0 \supseteq C_1 \supseteq C_2 \supseteq C_3 \supseteq \cdots$$

is a nested sequence of nonempty closed sets in X, then $\bigcap_{i=0}^{\infty} C_i \neq \emptyset$.

A2. Let X and Y be (disjoint) topological spaces, let A be a subspace of X, and let $g : A \to Y$ be a continuous function. Let $X \coprod Y$ be the union of X and Y with the disjoint union topology, and let $\sim$ be the equivalence relation on $X \coprod Y$ defined by $p \sim q$ if and only if either $p = q$ or $\{p, q\} = \{a, g(a)\}$ for some $a \in A$. Let $Z := (X \coprod Y)/\sim$ be the quotient space. Prove that there is an embedding $f : Y \to Z$ (i.e., a continuous function that is a homeomorphism onto its image).

A3. A space X is defined to be *totally disconnected* if the only nonempty connected subspaces of X are the subspaces of the form $\{p\}$ for some $p \in X$.

(a) Prove that if A is a subspace of a totally disconnected space X, then A is totally disconnected.

(b) Prove that if X is a totally disconnected space, then X satisfies the T_1 separation property.

A4. Prove that if X and Y are contractible spaces, then the product space $X \times Y$ is contractible.

Section B: Do THREE problems from this section.

B1. Let X be the quotient of the square $I \times I$ with respect to the equivalence relation $\sim$ on $I \times I$ generated by $(t, 0) \sim (1, t/2)$ for all $t \in I$ and $(0, 1) \sim (1, 1)$.

(a) Describe (draw) a CW complex structure for X.

(b) Use your CW structure to compute a presentation for the fundamental group $\pi_1(X)$, and show that $\pi_1(X)$ is isomorphic to a familiar group.

B2. A topological space X is *locally path-connected*, or *LPC*, if for every point $a \in X$ and every open set U of X containing a, there is an open set V of X such that $a \in V \subseteq U$ and (as a subspace of X) V is path-connected. Prove that if $p : Y \to X$ is a covering space and X is LPC, then Y is LPC.

B3. Let G be the free group on two generators a and b; that is, G is presented by $G = \langle a, b \mid \ \rangle$.

(a) Use covering space theory to prove that there is a finite index subgroup H of G such that $a^6 b^6$ is not an element of H.

(b) Use covering space theory to determine whether the following sentence is true (and prove your answer): For every natural number $n \geq 1$, there is a finite index subgroup H_n of G such that $a^n b^n$ is not an element of H_n.

B4. Prove directly from the definition of singular homology that if X is a path-connected topological space, then $H_0(X) \cong \mathbb{Z}$.