Math 871-872 Qualifying Exam

May 2021

Do three questions from Section A and three questions from Section B. You may work on as many questions as you wish, but you must indicate which ones you want graded. When in doubt about the wording of a problem or what results may be assumed without proof, ask for clarification. Do not interpret a problem in such a way that it becomes trivial.

Section A: Do THREE problems from this section.

A1. A topological space $(X, \mathcal{T})$ is called $T_{2.5}$ if for every pair of distinct points $a, b \in X$ there are open neighborhoods $U, V \in \mathcal{T}$ of a and b with disjoint closures: $\overline{U} \cap \overline{V} = \emptyset$. Show that the product $X \times Y$ of two $T_{2.5}$ spaces, with the product topology, is $T_{2.5}$.

A2. Let $(X, \mathcal{T})$ be a Hausdorff space and $* \notin X$. Let $Y = X \cup \{*\}$ and define

$$\mathcal{O} = \{\{*\} \cup (X \setminus C) \ : \ C \subseteq X \text{ is a compact subset of } X\}, \text{ and } \mathcal{T}' = \mathcal{T} \cup \mathcal{O}.$$

Show that $\mathcal{T}'$ is a topology on Y, and that, with this topology, Y is compact.

A3. Let $f : X \to Y$ and $g : Y \to X$ be continuous maps between topological spaces, with $f \circ g = \mathrm{Id}_Y$.

a) Prove that, if Y is path-connected and for every $y \in Y$ we have $f^{-1}(y)$ path-connected, then X is path-connected.

b) Prove the same statement with "path-connected" replaced by "connected" everywhere.

A4. Let $D = \{(x, y) \in \mathbb{R}^2 \ : \ x^2 + y^2 \leq 1\}$ be the unit disk with the usual topology, and let $\sim$ be the equivalence relation '$(x_1, y_1) \sim (x_2, y_2) \Leftrightarrow x_1 = x_2$' on D, and $q : D \to X = D/\sim$ the associated quotient map. Show that X, with the quotient topology, is homeomorphic to the interval $[-1, 1]$ (with the usual topology).

Section B: Do THREE problems from this section.

B1. Let X be the union of the top and bottom faces $[0,1]^2 \times \{0\}$, $[0,1]^2 \times \{1\}$ of the unit cube $C = [0,1]^3$ and the four 'main' diagonals of the cube - the line segments joining opposite vertices, $(0,0,0)$ to $(1,1,1)$, $(1,0,0)$ to $(0,1,1)$, $(0,1,0)$ to $(1,0,1)$, and $(0,0,1)$ to $(1,1,0)$. [The diagonals all therefore meet at $(1/2, 1/2, 1/2)$.] Compute a presentation for the fundamental group of X.

(You may treat X either as a CW complex or as a subspace of $\mathbb{R}^3$.)

B2. Let $p_1 : X_1 \to X$ and $p_2 : X_2 \to X$ be covering maps, and let $Y \subseteq X_1 \times X_2$ be defined as $Y = \{(x_1, x_2) \ : \ p_1(x_1) = p_2(x_2)\}$, with the subspace topology, and define maps $q_i : Y \to X_i$ by $q_i(x_1, x_2) = x_i$. Show that q_1 and q_2 are also both covering maps, and that $p_1 \circ q_1 = p_2 \circ q_2$.

B3. Let $X = S^1 \vee S^1$ be the bouquet of two circles, with the usual CW-structure and fundamental group $\pi_1(X, x_0) = F(a, b)$ generated by two loops, one around each circle. Let $H = \langle \ a \ , \ abab \ , \ bbabb \ , \ baab^{-1} \ , \ bbab^{-1} \ \rangle$ be the subgroup of $F(a, b)$ generated by these five elements. Construct a covering space $p : (\widetilde{X}, \widetilde{x}_0) \to (X, x_0)$ with $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0)) = H$. Based on your construction, is H a normal subgroup of $F(a, b)$?

B4. Let $\Delta_1 = [x_0, x_1, x_2]$ and $\Delta_2 = [y_0, y_1, y_2]$ be a pair of 2-simplices, and let X be the Δ-complex obtained from these simplices by identifying the pairs $\{x_0, y_0\}$, $\{x_1, y_1\}$, and $\{x_2, y_2\}$ together (i.e, $x_0 = y_0$, etc.). Describe the simplicial chain complex for X, and compute the simplicial homology groups of X.