Math 871-872 Qualifying Exam
January 2021

Do three questions from Section A and three questions from Section B. You may work on as many questions as you wish, but you must indicate which ones you want graded. When in doubt about the wording of a problem or what results may be assumed without proof, ask for clarification. Do not interpret a problem in such a way that it becomes trivial.

Section A: Do THREE problems from this section.

A1. Show that if $f : X \to Y$ is continuous and $A \subseteq X$ has $f(A) \subseteq Y$ closed, then $f(\overline{A}) = f(A)$.

A2. Suppose that $(X, \mathcal{T})$ is a topological space with the property that for every $x, y \in X$ with $x \neq y$ there is a continuous map $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ to a Hausdorff space (depending on x and y) so that $f(x) \neq f(y)$. Show that $(X, \mathcal{T})$ is Hausdorff.

A3. Given a function $f : X \to Y$ between metric spaces, define

$$\Gamma_f = \{(x, f(x)) : x \in X\} \subseteq X \times Y.$$

(a) Prove that if f is continuous then Γ_f is a closed set.

(b) Prove that if Y is compact and Γ_f is closed then f is continuous.

(c) Give an example of a discontinuous function $f : \mathbb{R} \to \mathbb{R}$ with Γ_f closed.

A4. Let $X = \mathbb{R}^2$ with the usual topology. Define

$$K_1 = \{(x, -x)\} \ : \ x \in \mathbb{R}\} \cup \Big(\bigcup_{a,b \in \mathbb{Q}} \{(a + x, b + x) \ : \ x \in \mathbb{R}\} \Big) \text{ and}$$

$$K_2 = \{(x, y) \in \mathbb{R}^2 \ : \ x \text{ and } y \text{ are either both rational or both irrational}\}.$$

Show that K_1 is a connected subset of X, and $K_1 \subseteq K_2$, and use this to show that K_2 is also a connected subset of X.

Section B: Do THREE problems from this section.

B1. Let X be the quotient space obtained from the annulus $S^1 \times [0, 1]$ (writing S^1 as the unit circle in the plane) by identifying $(x, 0)$ to $(-x, 0)$ and identifying $(x, 1)$ to $(-x, 1)$, for every $x \in S^1$. Find a presentation for the fundamental group of X, based at the point $x_0 = ((1, 0), 0)$.

B2. Show that if $p : \widetilde{X} \to X$ is a covering space, with X connected and locally path connected, $x_0 \in X$, and $x_1, x_2 \in p^{-1}(\{x_0\})$, then there is a deck transformation $h : \widetilde{X} \to \widetilde{X}$ with $h(x_1) = x_2$ if and only if $p_*(\pi_1(\widetilde{X}, x_1)) = p_*(\pi_1(\widetilde{X}, x_2))$, as subgroups of $\pi_1(X, x_0)$.

B3. Use covering spaces of the graph with one vertex and two edges to determine the index of the subgroup H of the free group $F(a, b)$ on two letters a, b which is generated by the elements $\{ab, b^2a^{-1}, b^{-1}a^2, b^2ab^2, ab^{-1}a^{-1}b\}$. Is H a normal subgroup of $F(a, b)$?

B4. Let X be the quotient space obtained from the unit square $[0, 1] \times [0, 1]$ by identifying the corner vertices $\{(0, 0), (0, 1), (1, 0), (1, 1)\}$ to a single point. Describe a Δ-complex structure for the space X and use it to compute the homology groups $H_i(X)$ of the space, for all i.