Math 871-872 Qualifying Exam
May 2015

Do three questions from Section A and three questions from Section B. You may work on as many questions as you wish, but indicate which ones you want graded. When in doubt about the wording of a problem or what results may be assumed without proof, ask for clarification. Do not interpret a problem in such a way that it becomes trivial.

Section A: Do THREE problems from this section.

(1) Let $X = \prod_{n\in\mathbb{Z}}[0,1] = \{(x_n)_{n\in\mathbb{Z}} \ : \ x_n \in [0,1]\}$, with each $[0,1]$ having the usual Euclidean topology and X having the product topology, and let

$$A = \{(x_n) \in X : \text{for some } N \text{ we have } x_n = 0 \text{ for all } n \geq N\} \ .$$

(a) Show that A is a dense subset of X, and is therefore <u>not</u> a closed subset of X.

(b) Show that, as a consequence, A is not a compact subset of X.

(2) If $X = \mathbb{R}^n$ with the usual Euclidean topology and $\mathcal{U} \subseteq X$ is an open, connected subspace of X, show that $\mathcal{U}$ is also path connected.

(3) Let $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ be a continuous map. A *section* of f is a continuous map $\sigma : (Y, \mathcal{T}') \to (X, \mathcal{T})$ so that $f \circ \sigma = \mathrm{Id}_Y$. Show that if f has a section, then f is a quotient map.

(4) Let $f : X \to Y$ be a continuous map, and suppose that there are continuous maps $g, h : Y \to X$ with $f \circ g \simeq \mathrm{Id}_Y$ and $h \circ f \simeq \mathrm{Id}_X$.

(a) Show that $g \simeq h$.

(b) Show, moreover, that f is a homotopy equivalence.

Section B: Do THREE problems from this section.

(5) Suppose X, X_1, X_2 are path-connected spaces, $p : X_1 \to X$ and $q : X_2 \to X$ are (surjective) covering maps, and $f : X_1 \to X_2$ is a continuous map with $q \circ f = p$. Show that f is also a covering map.

(6) Show, using covering space theory, that a finitely presented group G has only finitely many distinct subgroups $H \leq G$ of index 4.

(7) Find the (simplicial) homology groups of the space X obtained from the 3-simplex $\Delta^3 = [v_0, v_1, v_2, v_3]$ by gluing the face $[v_0, v_1, v_2]$ to the face $[v_0, v_2, v_3]$ and the face $[v_0, v_1, v_3]$ to the face $[v_1, v_2, v_3]$ (respecting the ordering of vertices as written (i.e., the first gluing map sends v_1 to v_2, and so on)).

(8) Compute the (reduced) singular homology groups of the space $X = S^1 \times (S^1 \vee S^1)$, which can be thought of as two copies of $S^1 \times S^1$ glued together along their copies of $S^1 \times \{x_0\}$. [You may use your knowledge of the homology groups of $T^2 = S^1 \times S^1$ in your calculations.]