Math 871-872 # UNL Math Topology Qualifying Exam **January 2017**

Do three questions from Section A and three questions from Section B. You may work on as many questions as you wish, but indicate which ones you want graded. When in doubt about the wording of a problem or what results may be assumed without proof, ask for clarification. Do not interpret a problem in such a way that it becomes trivial.

Section A: Do THREE problems from this section.

(1) Let X be the set of real numbers with the lower limit topology $\mathcal{T}$; that is, the topology generated by the basis $\{[a,b) \mid a,b \in X\}$. Let $Y = X \times X$ have the product topology.

(1a) Find the closure of the interval $(0,1)$ in X.

(1b) Determine whether Y is connected and/or compact.

(2) Let $X = \mathbb{R}^2$ have the Euclidean topology, let $C = (\mathbb{R} \times \{0\}) \cup (\{0\} \times \mathbb{R})$, and let $\mathcal{T}_1$ be the subspace topology on C. Define the function $f : X \to C$ by $f(p,q) = (p,0)$ if $p \neq 0$ and $f(p,q) = (0,q)$ if $p = 0$. Let $\mathcal{T}_2$ be the quotient topology on C induced by f. Determine whether $\mathcal{T}_1 = \mathcal{T}_2$.

(3) A space X is *completely regular* if X is T_1 and whenever C is closed in X and $p \in X \setminus C$, then there is a continuous function $f : X \to [0,1]$ such that $f(p) = 0$ and $f(C) = \{1\}$.

(3a) Let X be a T_1 space with topology $\mathcal{T}$ generated by a subbasis $\mathcal{S}$. Show that X is completely regular if and only if for every $p \in X$ and subbasis element $U \in \mathcal{S}$ containing p, there is a continuous function $f : X \to [0,1]$ such that $f(p) = 0$ and $f(X \setminus U) = \{1\}$.

(3b) Show that if X and Y are completely regular spaces, then so is the product space $X \times Y$. (Hint: Use the result in (3a).)

(4) A continuous function $f : X \to Y$ is called *proper* if for every compact subset C of Y, the preimage $f^{-1}(C)$ is compact in X. Show that if X and Y are compact Hausdorff spaces and $f : X \to Y$ is a proper continuous function, then f is a closed map.

Section B: Do THREE problems from this section.

(5) Let K be a graph with five vertices a_0, a_1, a_2, b_1, b_2 and six edges e_{ij} for $i \in \{0,1,2\}$ and $j \in \{1,2\}$, where e_{ij} has endpoints a_i and b_j. Let X be the CW complex obtained from K by attaching a 2-cell along each loop formed by a cycle of four distinct edges in K. Find $\pi_1(X)$ and show that $\pi_1(X)$ is isomorphic to a familiar group.

(6) Let $X = T^2 = (I \times I)/\sim$ (where $(x,0) \sim (x,1)$ and $(0,x) \sim (1,x)$ for all $x \in I$) be the torus with basepoint $x_0 = [(0,0)]$. Find two 2-sheeted covering spaces $p : (Y, y_0) \to (X, x_0)$ and $p' : (Y', y_0') \to (X, x_0)$ satisfying the property that $p_*(\pi_1(Y, y_0)) \neq p'_*(\pi_1(Y', y_0'))$.

(7) Let k and n be natural numbers greater than 1. Let G be a subgroup of index k of the free group F_n on n generators. Then G is isomorphic to the free group F_m for some natural number m; find m as a function of k and n.

(8) If X is a Δ-complex with at most one k-simplex for each $0 \leq k \leq 5$, show that each of the homology groups $H_k(X)$ must be cyclic for all $0 \leq k \leq 5$. Can all of these groups be non-trivial? Explain why or why not.