Math 871-872 Qualifying Exam
January 2019

Do three questions from Section A and three questions from Section B. You may work on as many questions as you wish, but you must indicate which ones you want graded. When in doubt about the wording of a problem or what results may be assumed without proof, ask for clarification. Do not interpret a problem in such a way that it becomes trivial.

Section A: Do THREE problems from this section.

(1) Let $\mathcal{T}, \mathcal{T}'$ be two topologies on X. Show that if $(X, \mathcal{T})$ is compact and Hausdorff, $\mathcal{T} \subseteq \mathcal{T}'$, and $\mathcal{T} \neq \mathcal{T}'$, then $(X, \mathcal{T}')$ is Hausdorff but **not** compact.

(2) Let $(X, \mathcal{T})$ be a path-connected space and $\mathcal{C} = \{U_\alpha\}_{\alpha \in I}$ an open covering of X. Show that for every $x, y \in X$ there is an $n \in \mathbb{N}$ and a sequence of sets $U_{\alpha_1}, \dots, U_{\alpha_n}$ in $\mathcal{C}$ with $x \in U_{\alpha_1}$, $y \in U_{\alpha_n}$ and $U_{\alpha_{i-1}} \cap U_{\alpha_i} \neq \emptyset$ for all $i = 2, \dots, n$.

(3) Let $X = \mathbb{R} \times \mathbb{R}$ with the product topology $\mathcal{T}$, giving $\mathbb{R}$ given the usual metric topology, and let $p : (X, \mathcal{T}) \to (\mathbb{R}, \text{usual})$ be the projection on the first coordinate, $p(x, y) = x$. Let $A = \{(x, y) \in X \ : \ x \geq 0 \text{ or } y = 0\}$, with the subspace topology that it inherits from X. Show that $f = p|_A : A \to \mathbb{R}$ is a quotient map, but that f is neither an open map nor a closed map.

(4) Show that if $f, g : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ and $h, k : (Y, \mathcal{T}') \to (Z, \mathcal{T}'')$ are continuous and $f \simeq g$ and $h \simeq k$, then $h \circ f \simeq k \circ g$. Show, however, that the converse ($h \circ f \simeq k \circ g$ implies $f \simeq g$ and $h \simeq k$) need not be true.

Section B: Do THREE problems from this section.

(5) Let $U = \{(x, y, 0) \in \mathbb{R}^3 \ : \ x^2 + y^2 = 1\}$, and let $W = \mathbb{R}^3 - U$.

Let $V = \{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 + y^2 + z^2 = 4\} \cup \{(0, 0, z) \in \mathbb{R}^3 \ : \ -2 \leq z \leq 2\}$.

(a) Describe a deformation retraction of W onto V.

(b) Use (a) and the Seifert-van Kampen Theorem to compute $\pi_1(W)$.

(6) Suppose $p : \widetilde{X} \to X$ is a covering space and X is Hausdorff. Prove that $\widetilde{X}$ is Hausdorff.

(7) Suppose $g : Y \to S^1$ is a path-connected 3-sheeted covering space. Prove that this covering space is unique up to isomorphism.

(8) Recalling that $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$, define the *equator* of S^2 to be $\{(x, y, z) \in S^2 : z = 0\}$, so that the equator is homeomorphic to S^1. Let Z_1 and Z_2 be disjoint copies of the 2-sphere S^2, let f be a homeomorphism from the equator of Z_1 to the equator of Z_2, and let Z be the quotient space obtained from $Z_1 \cup Z_2$ by identifying the equator of Z_1 to the equator of Z_2 via f. Find a Δ-complex structure on the space Z, and compute the simplicial homology groups of Z.