Math 871-872 Qualifying Exam
January 2023

Do three questions from Section A and three questions from Section B. You may work on as many questions as you wish, but you must indicate which ones you want graded.

Standard results my be used if the **statement of the result is clearly stated** and does not trivialize the problem. When in doubt about the wording of a problem or what results may be assumed without proof, ask for clarification. Do not interpret a problem in such a way that it becomes trivial.

Section A: Do THREE problems from this section.

A1. Suppose that X is compact, Y is Hausdorff, $f : X \to Y$ is a continuous surjective map, and $g : Y \to Z$ is a function. Show that if $g \circ f : X \to Z$ is continuous, then g is continuous.

A2. Let X and Y be nonempty topological spaces, and let $X \times Y$ have the product topology. Let $\sim$ be the equivalence relation on $X \times Y$ defined by $(x, y) \sim (x', y')$ if and only if $x = x'$. Prove that if $Z = (X \times Y)/\sim$ is the quotient space, then Z is homeomorphic to X.

A3. Let X be the set of real numbers, let $\mathcal{T}_E$ be the Euclidean topology on X, and let $\mathcal{T}_i$ be the included point topology on X (that is, $\mathcal{T}_i = \{U \subseteq X \mid 0 \in U\} \cup \{\emptyset\}$). For each of the following topological spaces, determine whether or not the space is connected.

(a) The set X with the topology $\mathcal{T}_E \cap \mathcal{T}_i$.

(b) The set X with the topology generated by the subbasis $\mathcal{T}_E \cup \mathcal{T}_i$.

A4. Let A be a subspace of a topological space X. A *weak deformation retraction* of X onto A is defined to be a continuous function $H : X \times [0, 1] \to X$ satisfying

(i) $H(x, 0) = x$ for all $x \in X$,
(ii) $H(x, 1) \in A$ for all $x \in X$, and
(iii) $H(a, t) \in A$ for all $a \in A$ and $t \in [0, 1]$.

Show that if there is a weak deformation retraction of X onto A, then X is homotopy equivalent to A.

Section B: Do THREE problems from this section.

B1. Let X be the CW-complex obtained from the 1-skeleton Γ of the 3-simplex $\Delta = [x_0, x_1, x_2, x_3]$ by attaching a 2-cell along a loop that traverses four edges in Γ, from x_0 to x_1 to x_2 to x_3 to x_0. Find a presentation for $\pi_1(X, x_0)$, and show that it is isomorphic to a familiar group.

B2. Let X and Y be path-connected spaces and let $p : Y \to X$ be a covering space. Suppose that $x_0 \in X$ and $y_0, y_1 \in p^{-1}(\{x_0\})$. Let $G = \pi_1(X, x_0)$ and for each $i \in \{0, 1\}$ let $H_i = p_*(\pi_1(Y, y_i))$. Show that there is an element $g \in G$ such that $gH_0g^{-1} \subseteq H_1$.

B3. Use covering space theory to find generating sets for two non-conjugate subgroups of index 3 in the free group $F(a, b)$ on two letters.

B4. Let $\Delta_1 = [x_0, x_1, x_2]$ and $\Delta_2 = [y_0, y_1, y_2]$ be a pair of 2-simplices, and let X be the Δ-complex obtained from these simplices by identifying the (directed) edge $[x_0, x_1]$ with $[y_0, y_1]$, and identifying the vertex x_2 with y_2. Describe the simplicial chain complex for X, and compute the simplicial homology groups of X.