Math 871-872 Qualifying Exam
June 2023

Do three questions from Section A and three questions from Section B. You may work on as many questions as you wish, but you must indicate which ones you want graded.

Standard results my be used if the **statement of the result is clearly stated** and does not trivialize the problem. When in doubt about the wording of a problem or what results may be assumed without proof, ask for clarification. Do not interpret a problem in such a way that it becomes trivial.

Section A: Do THREE problems from this section.

A1. A topological space $(X, \mathcal{T})$ is called *clumped* if for every $U, V \in \mathcal{T}$ with $U \neq \emptyset \neq V$ we have $U \cap V \neq \emptyset$. Show that if X and Y are both clumped spaces, then the space $X \times Y$, with the product topology, is clumped.

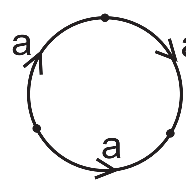
A2. Suppose that $(X, \mathcal{T})$ is a Hausdorff space, and $A \subseteq X$ is a compact subspace of X. If $p \in X$ with $p \notin A$, show that there are open sets $U, V \in \mathcal{T}$ with $p \in U$, $A \subseteq V$, and $U \cap V = \emptyset$.

A3. For any topological space $(X, \mathcal{T})$ and subset $A \subseteq X$, the *boundary* of A in X, denoted $Bd_X A$, is defined as $Bd_X A = Cl_X(A) \cap Cl_X(X \setminus A)$ (where $Cl_X(A)$ denotes the closure of A in X). Show that if $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is continuous and $B \subseteq Y$, then $Bd_X(f^{-1}(B)) \subseteq f^{-1}(Bd_Y(B))$.

A4. Suppose that X is a topological space and $f_1, f_2 : S^1 \to X$ are homotopic continuous maps. For each $i \in \{1, 2\}$, let Z_i be the quotient space $Z_i = \left(X \coprod D^2\right) / \sim_i$ where $\sim_i$ is the equivalence relation generated by $x \sim_i f_i(x)$ for all $x \in S^1 = \partial D^2$ (that is, Z_i is X with a 2-disk attached along its boundary, via the attaching map f_i).

(a) Find a single space Z such that for each $i \in \{1, 2\}$, the space Z contains, and deformation retracts to, the space Z_i. (You only need to give the space Z; you do not need to prove that your space Z has these properties.) Explain why this shows that the spaces Z_i built by attaching a 2-disk to X along homotopic attaching maps are homotopy equivalent.

(b) Let Y be the quotient of the disk D^2 given by the identifications on the boundary of D^2 in the figure below. Show that Y is contractible.



Section B: Do THREE problems from this section.

B1. Let x_0 be a point in S^1, and let X be the quotient space obtained from the torus $T = S^1 \times S^1$ by identifying (x, x_0) with $(-x, x_0)$ for all $x \in S^1$. Starting from the representation of the torus T as the quotient of a square $[0, 1] \times [0, 1]$ with edges identified, describe the quotient space X as a 2-disk with identifications along its boundary, and find a presentation for the fundamental group of X.

B2. Suppose that $p_1 : (\widetilde{X}_1, \widetilde{x}_1) \to (X, x_0)$ and $p_2 : (\widetilde{X}_2, \widetilde{x}_2) \to (X, x_0)$ are covering maps satisfying that $\widetilde{X}_1$, $\widetilde{X}_2$, and X are path connected, locally path connected, and semi-locally simply connected. Show that there exist a space $\widetilde{X}$ with a basepoint $\widetilde{x} \in \widetilde{X}$, and covering maps $q_i : (\widetilde{X}, \widetilde{x}) \to (\widetilde{X}_i, \widetilde{x}_i)$ for $i = 1, 2$ satisfying the properties that $p_1 \circ q_1 = p_2 \circ q_2$ and a loop $\gamma : (I, \partial I) \to (X, x_0)$ lifts to a loop in $(\widetilde{X}, \widetilde{x})$ (under $p_1 \circ q_1$) if and only if γ lifts to a loop under <u>both</u> p_1 and p_2.

B3. Let $X = S^1 \vee S^1$ be the bouquet of two circles, with the usual CW-structure and fundamental group $\pi_1(X, x_0) = F(a, b)$ generated by two loops, one around each circle. Let $H = \langle ab, ba, aaaa, bbb \rangle$. Build the covering space $p : \widetilde{X} \to X$ with $p_*(\pi_1(\widetilde{X}, \widetilde{x}_0) = H$ (be sure to label your chosen basepoint $\widetilde{x}_0$). What is the index of H in $F(a, b)$? Is H a normal subgroup of $F(a, b)$?

B4. Let X be the union of the 1-skeleton of the 3-simplex Δ^3, with vertices v_0, v_1, v_2, v_3, and the two 2-simplices $[v_0, v_1, v_2]$ and $[v_0, v_1, v_3]$, with a Δ-complex structure given by the 0-, 1-, and 2-simplices from Δ^3. Find the simplicial chain groups and boundary maps, and using your simplicial chain complex, compute the simplicial homology groups of X.