Math 871-872 Qualifying Exam
May 2018

Do three questions from Section A and three questions from Section B. You may work on as many questions as you wish, but you must indicate which ones you want graded. When in doubt about the wording of a problem or what results may be assumed without proof, ask for clarification. Do not interpret a problem in such a way that it becomes trivial.

Section A: Do THREE problems from this section.

(1) Show that if $(X_\alpha, \mathcal{T}_\alpha)$ are path-connected spaces, for $\alpha \in I$, then $\prod_{\alpha \in I} X_\alpha$, with the product topology, is also path-connected. Show, on the other hand, that there are examples where this is false, if we use the box topology instead.

(2) Recall that, for $A \subseteq X$, the set X/A is the set of equivalence classes under the relation $x \sim y$ iff $x = y$ or $x, y \in A$. Show that if X is the real line $\mathbb{R}$ (with the usual topology) and $A = (-\infty, 0) \cup (1, \infty) \subseteq \mathbb{R}$, that X/A, with the quotient topology, is compact but <u>not</u> Hausdorff.

(3) Show that a closed subset $A \subseteq X$ of a normal (i.e., T_1 and T_4) space $(X, \mathcal{T})$, with the subspace topology, is normal.

(4) Show that if X and Y are topological spaces, and $X \times Y$, with the product topology, is contractible, then both X and Y are contractible.

Section B: Do THREE problems from this section.

(5) Let X be a path-connected space containing points x_0 and x_1, and let P be the set of path-homotopy classes of paths from x_0 to x_1 in X. In other words, elements of P are equivalence classes of paths from x_0 to x_1, where two paths h_0 and h_1 are equivalent if there exists a homotopy $h_t : [0, 1] \to X$ from h_0 to h_1 such that $h_t(0) = x_0$ and $h_t(1) = x_1$ for all $t \in [0, 1]$. Prove that there is a bijection from P to $\pi_1(X, x_0)$.

(6) A topological space X is called *locally Euclidean* if every point $x \in X$ has an open neighborhood U such that U is homeomorphic to an open set in $\mathbb{R}^n$ for some n. Prove that if X is locally Euclidean and $p : \widetilde{X} \to X$ is a covering space, then $\widetilde{X}$ is locally Euclidean.

(7) Let F_n denote the free group on n letters. Use covering space theory to show that for all n, there exists a subgroup H_n of F_2 such that $H_n \cong F_n$. For your choices of H_n constructed, determine if each H_n is normal in F_2.

(8) Let Δ_2 and Δ_2' be distinct 2-simplices, and let X be the quotient space obtained by identifying the six vertices of $\Delta_2 \cup \Delta_2'$ to a single point. Identify a Δ-complex structure for X and compute the simplicial homology groups $H_n^\Delta(X)$ for all n.