Course Materials

Math in the City Workshop

University of Nebraska-Lincoln

December 3-4, 2010

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Contents

1 Introduction

This set of notes is the material that we covered during one semester over 10-12 lectures in Math in the City. During this semester, our projects dealt with issues facing recycling companies, but this material could also be useful for sustainable design projects

The material is organized as follows: first, we present a rough outline of the semester, to give an idea how the course was structured. All sections after that deal with the mathematical content covered in the course, followed by an appendix which is referenced throughout. Each section contains a set of problems which can serve as further examples during class or homework problems.

2 Outline of the Beginning of the Semester

Day 1: Pass out the syllabus, talk about the structure of the course. Present Ideas for projects, talk about information you have already gathered, what you think the project will entail, etc. - give student survey (included in the appendix)

Day 2: Present Assignment 1. This assignment focuses on gathering background information relevant to the assigned projects. See attached Homework 1. Begin presenting ideas about order of magnitude and making reasonable estimations, see Content, Day 2 below.

Day 3: Meet as a class and go through the data sets and discuss more background about the projects. This is not the data that was assigned for the students to gather in Assignment 1, rather, it is data and information you have already gathered from your contacts in the industry. Inform your students how to read the data you have gathered (this may or may not be straightforward) and also relay relevant information that you gathered during these meetings. At the end of class, have the students split into their groups to begin splitting up tasks that they need to accomplish to start their projects. For example, you may assign one person to look at the data and compile it in a more user-friendly format. Also, if there are other relevant pieces of information to gather that was not covered in the more general Assignment 1, you may assign different students to gather this information.

Day 4: Beginning of week 2, after the students have had a weekend to look over the data, you could talk very in depth about the background information on the industry you are researching. We actually had a guest speaker from a recycling company talk about how the company operates. Topics covered included, but were not limited to: operating costs, budget, amount of materials collected per year, and breakdown in revenue per material.

Day 5: See Content, Day 5. Continue lecture on mathematical modeling.

Day 6: See Content, Day 6. After lecture, meet in groups to discuss what data needs to be collected by next week's group meetings. For example, one person should find the times and distances from location to location, one could find the price paid per commodity, etc.

Day 7: Pass out Homework 2, more problems inspired by previous lectures. Could actually be taken from exercises in "Consider a Spherical Cow." See appendix. Also, begin lecturing on Linear Programming. See Content, Day 7.

Day 8: Hold group meetings. Begin finding coefficients for cost, benefit, and time functions. Clarify what data you have gathered already and what data you still need to gather.

Day 9-10: Continue lecture on Linear Programming. See Content, Day 9-10.

Day 11: Introduction to Sage Programming Language. See Appendix.

Day 12-13: Continue lecture on Linear Programming. See Content, Day 12-13.

Day 14-18: During this semester, the students prepared to present some of their preliminary results at a conference midway through the semester. These days were spent entirely on finishing up the preliminary results, fitting the data we had into a model, and drawing conclusions from this model. There was also a poster session during this conference, so we spent time making the poster and practicing presenting them. All of this was done not only during class time, but in individual meetings of each group with the instructors of the course. Also, during this time assign the students to complete a "memo" of their project so far. This is simply a summary of their results so far

Day 19-20: Pass out Homework 3, concerning line of best fit. Also, begin lecture on integer programming, and a brief introduction to graph theory, see Content, Day 19-20.

Day 22: Lecture on the traveling salesman problem, see Content, Day 22.

Day 23: Introduction to programming the traveling salesman problem into Sage. See Appendix.

Day 24 onward: At this point in the semester, most of the time should be spent on creating mathematical models for your data and programming those into whatever mathematical software you choose to use. Since the traveling salesman problem can be very computationally intensive, and thinking about how to make the problem fit with each individual scenario can be difficult, we began with a very basic approach, trying to find one individual tour of all of the sites together. After this, we spent a significant amount of time brainstorming how to adapt our model to fit the actual constraints that the companies were facing. In particular, we had to think about how to divide the individual sites into multiple tours, one for each day of the work week.

Course Content

3 Mathematical Modeling

3.1 Day 2: Guesstimates

The goal of this section is to give students a glimpse of how different the problems in this class are from a standard mathematical course. Although the initial problems require no advanced mathematical techniques, students find the topics intriguing, and sometimes even challenging. The book by J. Harte, "Consider a Spherical Cow" is an excellent resource for these first lectures.

Question: How many plumbers are there in the U.S.?

Let us begin by brainstorming how we may estimate this. There are several options: use the phonebook and count how many plumbers are in your town, then extrapolate to the population of the U.S.; contact a tax office for plumbing businesses; search on the internet for Census Bureau Statistics. The point of this exercise though is to practice making reasonable assumptions to come up with rough estimates. By rough estimates we mean to find the order of magnitude for these numbers.

▷First Approach:

The first approach relies on the fundamental assumption that the number of plumbers will be determined by demand in the marketplace. Now we just need to determine approximately how much demand there is for plumbers.

Suppose (based purely on personal experience, but this seems reasonable) that

(A1) each household needs a plumber approximately once every two years.

Also suppose that

(A2) a typical job takes about 1 hour to complete.

Now, we simply need to estimate the number of households in the U.S. We know that the population of the country is about 300 million. So if we assume that there are 5 people per household, we get that there are $\frac{300}{5} = 60$ million households in the United States. (This seems to be exaggerating the number of people per household, and in fact a quick internet search reveals that there are at least 115 million households in the U.S. with a television, so 60 million is probably about half as large as it should be. However, the point of this exercise is to find the order of magnitude and making this change will not change our final conclusion.) So:

$$\left(\frac{1\text{Hour of work}}{2\text{year} \cdot \text{Household}}\right) \cdot \left(60 \cdot 10^6 \text{Households}\right) = 30 \cdot 10^6 \frac{\text{Hours of work}}{\text{year}}$$

Now, if we suppose that each plumber works (on the actual job site, excluding time for transit, preparation, and other miscellaneous tasks) 4 hours per day, 5 days per week for 50 weeks per year, this provides our estimate that each plumber will work

$$\left(5\frac{\text{days}}{\text{week}}\right) \cdot \left(50\frac{\text{weeks}}{\text{year}}\right) \cdot \left(4\frac{\text{hours}}{\text{day}}\right) = 1000\frac{\text{hours}}{\text{year}}.$$

Therefore:

$$\text{Number of Plumbers} = \frac{30 \cdot 10^6 \frac{\text{hours}}{\text{year}}}{1000\frac{\text{hours}}{\text{year}\cdot\text{plumber}}} = 30 \cdot 10^3 \text{plumbers}$$

It should be noted here that this only accounts for residential plumbers, not accounting for those who serve businesses or government buildings. Furthermore, changing a lot of these assumptions by small factors will not change our basic conclusion that there are somewhere on the order of 10^4 plumbers in the U.S. For example, if we replace 60 million households with 120 million, our final conclusion is that there will be around 60,000 plumbers, which is again order of magnitude 10^4.

▷Second Approach:

Let us try this time to estimate the number of plumbers by using their total income. We begin with similar assumptions, that there will be one job every 2 years per household and that the typical job takes 1 hour to complete. However, this estimate relies on the cost, so we will assume that the average job costs $ 150.

Now, we use our same estimate from last time that there are about $30 \cdot 10^6$ jobs per year in the entire country, so the total amount spent on plumbers in the entire country per year is:

$$\frac{\$150}{\text{job}} \cdot [30 \cdot 10^6 \text{jobs}] = \$4500 \cdot 10^6.$$

Now we can estimate how much we think the typical plumber makes in one year. Suppose that the typical plumber makes $ 50,000 per year. Then:

$$\text{Plumbers} = \frac{\$4500 \cdot 10^6}{\frac{\$50 \cdot 10^3}{\text{plumber}}} = 90 \cdot 10^3 \text{plumbers}.$$

Even though in magnitude the two estimates are fairly far apart, we still come to the conclusion that there are somewhere on the order of 10^4 [residential] plumbers in the country.

3.2 Day 5-6: Modeling problems; exponential population growth

Once again this lecture is inspired by J. Harte, "Consider a Spherical Cow."

Question: If the global human population continues to grow at the rate it averaged between 1950 and 2000, how long will it take for the average human population density on all Earth to equal the 2000 population density in typical urban areas of the world?

To solve this, we will use the most simple population growth model: the natural growth or exponential growth model. If $N(t)$ denotes the world population at time t, then we assume $N' = rN$. Basically, the rate that a population is growing at any point in time ought to be proportional to the population at that point. Say $N(0)$ is the population in 1950, and t is measured in years. An easy way to find the solution of this equation is to observe:

$$\frac{N'(t)}{N(t)} = r$$

$$\Rightarrow \int \frac{N'(t)}{N(t)}\mathrm{d}t = \int r\,\mathrm{d}t$$

$$\Rightarrow \ln N(t) = rt + c$$

$$\Rightarrow N(t) = e^{rt}e^{c}$$

Since c is an arbitrary constant, we could express the final solution as $N(t) = ce^{rt}$. Now, if we set $t = 0$, we get $c = N(0)$, so $N(t) = N(0)e^{rt}$.

Now, a bit of background research reveals that the world population in 1950 was about 2.5 billion, while in 2000 it was 6.1 billion. Therefore, we know $N(0) = 2.5 \cdot 10^9$ and $N(50) = 6.1 \cdot 10^9$. We can now use this information to find r, since

$$N(50) = N(0)e^{50r} \quad \Rightarrow \quad r = \frac{1}{50}\ln\left(\frac{N(50)}{N(0)}\right) \approx .017$$

Here, we include a brief discussion about how to interpret this model. Since the growth rate is 1.7%, does that mean that at the end of a calendar year the new population will be 1.7% higher than it was at the beginning of the year? Technically, the answer is no, if we recall, for example, exponential models that deal with compound interest. Having an exponential model represents continuously compounded interest, which yields a higher percentage gained at the end of the year.

However, we also recall from calculus that the first two terms in the Taylor series expansion for e^x are $1 + x$, so if we have $0 < x << 1$, then $e^x \approx 1 + x$. Therefore, since our growth rate is fairly small, it actually is pretty accurate to say that at the end of one calendar year the population will be 1.7% higher than it was at the beginning of the year.

Now that we have our population model, it is not difficult to derive a density model, since density is simply population divided by area. Again, a quick internet search gives us that the land area on the entire planet is about $1.5 \cdot 10^8$ km^2. This allows us to define a population density function, $D(t)$ by

$$D(t) = \frac{1}{1.5 \cdot 10^8}N(t) = \frac{2.5 \cdot 10^9}{1.5 \cdot 10^8}e^{.017t}.$$

Now, to finish the question we need to determine the population density of the city that we asked about in the statement of the problem. Let's assume that we are talking about a city with 10 million people, in an area of 1000 square kilometers. This city would have a density of 10^4 people per square kilometer (this is actually a fairly close approximation of the density of New York City). Therefore, find the time T at which the world population density reaches this point:

$$10^4 = D(T) = \frac{50}{3}e^{.017T} \quad \Rightarrow \quad T = \frac{1}{.017}\ln 600 \approx 376.$$

So if we assume that the world population continues to grow at the average rate that it grew between 1950 and 2000, then it will take about 376 years from 1950, or the year 2326 for the world population density to reach that of New York City.

3.3 Day 5-6: Modeling Problems: Urban Heat Islands

Question: How much warmer is it in a city compared to the countryside as a result of population density?

To begin, we know that the average temperature of the earth is approximately 290 Kelvin, or about $17°C$. For simplicity, we will assume that heat flow is determined primarily by a horizontal convective process.

Consider a square urban area, 20 km on each side, with a population approximately 10^7. Let's further assume that residents consume energy at the average rate per capita for the U.S. in 1980, i.e., somewhere on the order of $11.2 \cdot 10^3 \frac{\text{watts}}{\text{person}}$.

Given this assumption, the amount of power per square meter in the city generated by people can be measured, say:

$$W = \frac{11.2 \cdot 10^3 \frac{\text{watts}}{\text{person}} \cdot 10^7 \text{people}}{(20\,\text{km})^2} = 280 \frac{\text{watts}}{\text{m}^2}.$$

Now, we are assuming a horizontal convective heat flow, so what will move heat out of the city are winds. We assume, on average, there will be wind flowing at about 10 kilometers per hour perpendicular to one of the edges of the city.

With these assumptions, we can calculate the residence time for heat in this system, it is simply $\frac{10\,\text{km}}{10\frac{\text{km}}{\text{hour}}} = 1\,\text{hour}$. Therefore, the total anthropogenic heat for the city H_a is given by:

$$\begin{aligned} H_a &= (\text{Residence Time}) \cdot (\text{Total inflow rate}) \\ &= (1\,\text{hour}) \cdot \left(280 \frac{\text{watts}}{\text{m}^2}\right) \cdot (20\,\text{km})^2 \\ &= 4 \cdot 10^1 4\,\text{J} \end{aligned}$$

Now, before we can calculate the temperature change that this causes, we first need to find the total mass of air in the city. We already know the length and width of the city, but what should we take the height to be? Of course, this depends on the level of cloud cover and smog, but we can guess on an average day that it will be about 300 meters. Therefore, the total mass of air in the city is:

$$M_a = (20\,\text{km})^2 \cdot (.3\,\text{km}) \cdot \left(1290 \frac{\text{g}}{\text{m}^3}\right) = 1.5 \cdot 10^1 4\,\text{g}.$$

Please note, we had to look up the density of air (1290 grams per cubic meter) and also convert the units of some numbers, but those steps were excluded. Finally, once we note that c_ρ, the specific heat of air is 1 Joule per gram degree Celsius, we can find the heat index:

$$\begin{aligned}\Delta T &= \frac{H_a}{c_\rho \cdot M_a} \\ &= \frac{4 \cdot 10^{14}\,\mathrm{J}}{\frac{1\,\mathrm{J}}{\mathrm{g}\,{}^\circ\mathrm{C}} \cdot 1.5 \cdot 10^{14}\,\mathrm{g}} \\ &= 3^\circ\mathrm{C}\end{aligned}$$

3.4 Problems and exercises

4 Day 7, 9-10, 12-13: Introduction to Linear Programming

This is the first bit of mathematical content covered that was directly relevant to the projects for this semester. For the projects dealing with recycling, we initially considered a much simplified model, where each truck will need to leave from a central location, pick up the commodities at this location, and then return to the central location. Therefore, for each location we will have a variable, call it x_i, which will represent how many times we want to visit that location. Associated with each variable will be a cost function, c_i, and a benefit function, b_i. Very simply, we would like to minimize costs while maximizing benefits, subject to certain real world constraints.

Typically, this could be thought of as a sort of calculus problem, where constraints are introduced using Lagrange multipliers and we optimize by taking derivatives and setting equal to zero. However, when the problem is linear everything simplifies quite a bit and we can approach the problem using linear programming. Most of the content from this section was taken from Hastings' "Introduction to the Mathematics of Operations Research with Mathematica."

We begin our discussion with an example. Suppose we have a winery which makes 3 types of wine; red, white and rose. The ingredients for the wine are two different types of grapes and sugar, there is also a certain number of labor hours to produce each gallon of wine and a profit per gallon for the finished product. This information is summarized in the following table.

Wine	$ per gallon	type 1 grapes: bushels per gallon	type 2 grapes: bushels per gallon	sugar: lbs per gallon	labor hours: per gallon
Red	1.25	2	0	2	2
White	1.50	0	2	1	1
Rose	2.00	1	1	1.5	2

Now, we ask how much wine should we produce each week to maximize profit? To start to answer this question, define 3 variables, x_1, x_2, and x_3, which represent the gallons of red, white, and rose wine produced each week, respectively.

Therefore, our total weekly profit will be given by: $1.25x_1 + 1.5x_2 + 2x_3$. We want to maximize this function.

However, there are several constraints given by our situation. The first is obvious, (1) $x_i \geq 0$, for $i = 1, 2, 3$. The other constraints we assume are:

(2) We may use no more than 200 bushels of type 1 grapes per week. Or, $2x_1 + x_3 \leq 200$.

(3) $2x_2 + x_3 \leq 150$. No more than 150 bushels of type 2 grapes per week.

(4) $2x_1 + x_2 + 1.5x_3 \leq 90$. No more than 90 lbs of sugar per week.

(5) $2x_1 + x_2 + 2x_3 \leq 250$. No more than 250 hours of labor per week.

We can write this problem in a different form. First, define $c, x \in \mathbb{R}^3$ by $x = (x_1, x_2, x_3)$ and $c = (1.25, 1.5, 2)$. Also, given $y, z \in \mathbb{R}^n$ we say that $y \leq z$ if the inequality is true for each component of y and z. With this in mind, define the matrix A by

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 2 & 1 & 1.5 \\ 2 & 1 & 2 \end{pmatrix}$$

and a vector b by $b = (200, 150, 90, 250)$. Now our problem can be expressed more succinctly as:

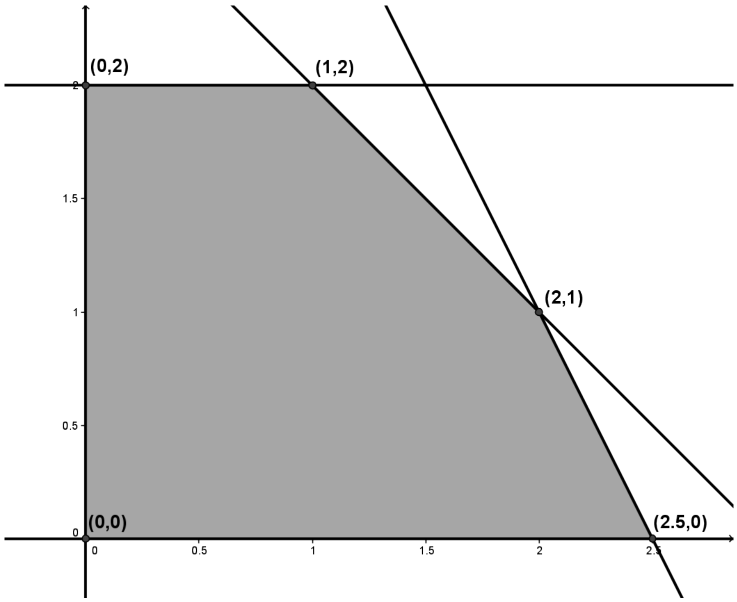
Maximize $f(x) = c \cdot x$

Subject to constraints $\begin{cases} Ax \leq b \\ x \geq 0 \end{cases}$

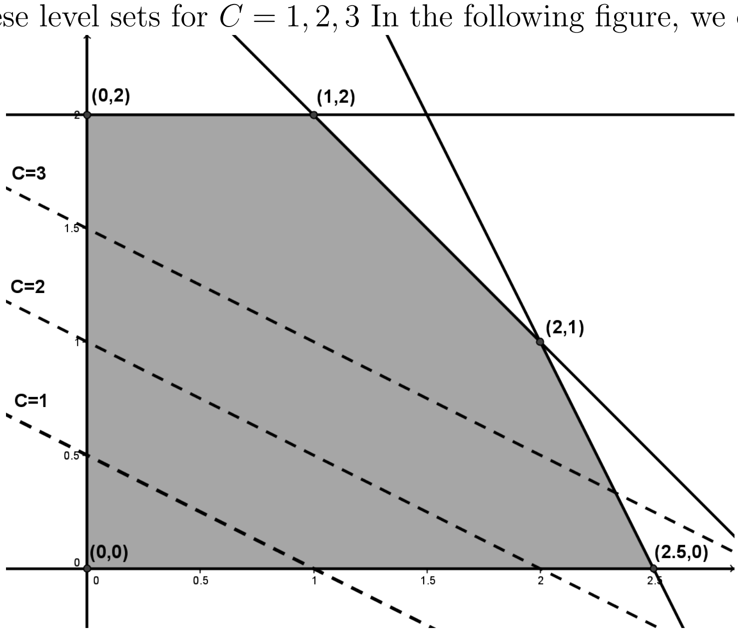
Now that we understand the process of setting up these problems let's look at a simpler problem to begin thinking about how to solve these.

This time, we do not include a story, let's just say we want to maximize $f(x_1, x_2) = x_1 + 2x_2$ (which we call the <u>objective function</u>). Subject to constraints: $\begin{cases} x_2 \leq 2 \\ x_1 + x_2 \leq 3 \\ 2x_1 + x_2 \leq 5 \\ x_1, x_2 \geq 0 \end{cases}$

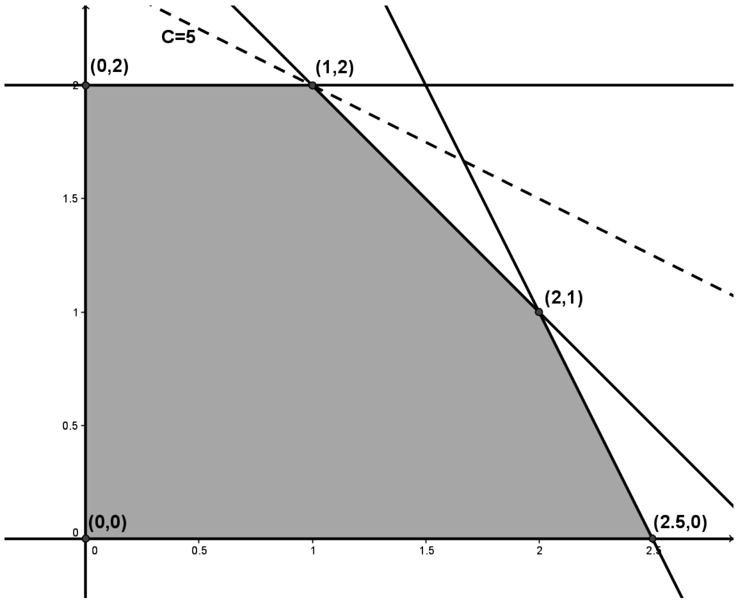
Since we now have a problem with only two variables, we can visualize all of the constraints in the $x_1 - x_2$ plane.



To begin thinking about this, we can draw the level sets of the function, i.e., graph the set of all points where $f(x_1, x_2) = C$, for some constant C. We can begin visualizing this by graphing these level sets for $C = 1, 2, 3$ In the following figure, we can see the graph of these level sets:



It is important to emphasize how to interpret the level sets. By saying $C = 1$, what we mean is that along the entire line $x_1 + 2x_2 = 1$, we have that $f = 1$. Since part of that line intersects the feasible region, we know that there are values of (x_1, x_2) for which $f(x_1, x_2) = 1$ while meeting all of the constraints. However, the fact that $C = 3$ intersects the feasible region means that we can do better. Note if $C = 5$, we get the level curve intersecting the region in only one point:



This leads us to believe that $(x_1, x_2) = (1, 2)$ will maximize f and that 5 is the largest possible value of f subject to the constraints. In fact this is the case, though we will explain more later how to be certain of this conclusion.

To summarize, there are three steps to solving a two dimensional problem like this geometrically.

1. Sketch the feasible region.

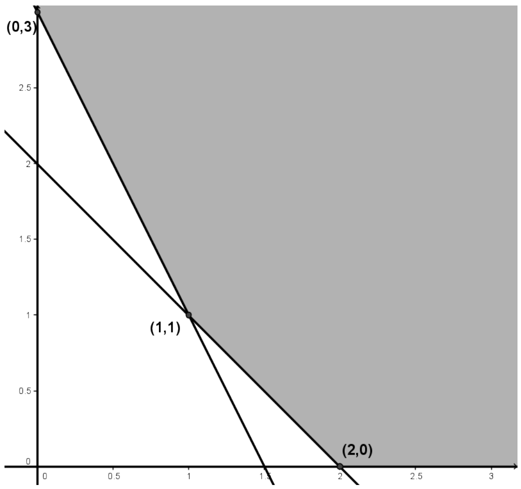
2. Sketch the level curves.

3. Identify the intersection point(s) between the highest level and the feasible region. Note: if the cost function has the same slope in the x_1x_2 plane as one of the constraints, then there may be infinitely many points where the function is maximized. For example, in the last problem if we have $f(x_1, x_2) = x_1 + x_2$ then the solution would be attained on the whole line segment between (1,2) and (2,1).

END OF DAY 7, BEGINNING OF DAY 9

We could also ask another type of question, what if our feasible region is unbounded? Consider a problem where our constraints are: $\begin{cases} 2x_1 + x_2 \geq 3 \\ x_1 + x_2 \geq 2 \\ x_1, x_2 \geq 0 \end{cases}$

In this case, the feasible region is unbounded:



In this case, which of the following three problems would have a solution?

- Minimize $f_1(x_1, x_2) = 3x_1 + 2x_2$
- Minimize $f_2(x_1, x_2) = x_1 - x_2$
- Maximize $f_3(x_1, x_2) = x_1 + 3x_2$

For f_1, note that f_1 will become unboundedly large if either x_1 or $x_2 \to \infty$ within the feasible region. Since we are looking for a minimum value of f_1 we can still test the corners of the region. If we do that we see f_1 attains its minimum at $(1, 1)$, taking on a value of 5.

With f_2, if we fix x_1 and let $x_2 \to \infty$ in our region, then $f_2 \to -\infty$. Therefore, f_2 has no minimum in our region.

Finally, if we consider f_3, we note that if either x_1 or $x_2 \to \infty$, then $f_3 \to \infty$. This tells us that f_3 has no maximum in our feasible region.

Note that both problems involving f_2 and f_3 have no solution because of how each particular function grows unboundedly large (either in the positive or negative direction) in the feasible region. For this reason, we call these unbounded problems. Note that it is not the feasible region alone which determines whether or not we have an unbounded problem. Additionally, since every problem we have in this topic is linear, we cannot have an unbounded problem if we have a bounded feasible region.

Here, we should note that another possible problem could arise if the constraints are inconsistent.

For example, if we have constraints: $\begin{cases} x_1 + x_2 \leq 1 \\ 4x_1 + 2x_2 \geq 8 \\ x_1, x_2 \geq 0 \end{cases}$

It is easily checked by graphing the constraints that there are no values of x_1 and x_2 which can satisfy all of these constraints, so regardless of the objective function this problem would not have a solution. With these considerations in mind, we can slightly modify the methodology for solving these problems:

1. Sketch the feasible region (if it is empty, there is no solution).

2. If the feasible region is unbounded, examine the objective function to see if the problem is unbounded.

3. If f is bounded, then at least one solution exists (there may be infinitely many). The solution will be taken on at a corner of the feasible region. One may intersect the c-level sets of f with the feasible region to determine the values f takes at each corner.

4. If two corner points both achieve the same optimal value for f, then so do all the points on the line segment that joins them.

The last item is true because of the fact that the feasible region of any linear programming problem is the intersection of half planes, and is therefore convex. Recall: a set S is convex if for all $x, y \in S$ the line segment joining x and y is also completely contained in S.

Also recall the definition of a convex combination. Let $x_1, \ldots, x_n \in \mathbb{R}^m$. Then a convex combination of these points is a vector $x = c_1x_1 + \ldots c_nx_n$ where $c_j \in [0, 1]$ for all j and $c_1 + \ldots c_n = 1$.

It is worth noting a couple facts at this point. First, any point in a bounded feasible region can be expressed as a convex combination of corner points (also called extreme points). Second, the corner points do not lie on any line segment connecting two other feasible points.

To see an example of how to write a point in the feasible region as a convex combination of the extreme points, consider the feasible region of problem 2. We have $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is in the region, so to write it as a convex combination of the corners we would need:

$$\begin{cases} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 0 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_4 \begin{bmatrix} \frac{5}{2} \\ 0 \end{bmatrix} + c_5 \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \sum_{j=1}^{5} c_j = 1 \end{cases}$$

END OF DAY 9, BEGINNING OF DAY 10

Here, we have 3 equations and 5 unknowns, which implies we have infinitely many solutions. If we fix c_5 and c_4, then solve for the rest of the variables in terms of these two, we derive the system:

$$\begin{cases} c_1 = 1 - \frac{1}{2}c_4 - 3c_5 \\ c_2 = -1 + \frac{3}{2}c_4 + 4c_5 \\ c_3 = 1 - 2c_4 - 2c_5 \\ c_4 = c_4 \\ c_5 = c_5 \end{cases}$$

Of course there are infinitely many choices for c_5 and c_4. If, for example, we choose $c_4 = c_5 = \frac{1}{4}$, then we can see, from the other equations we get $c_1 = \frac{1}{8}$, $c_2 = \frac{3}{8}$, $c_3 = 0$.

Now, let's return to problem 2. The most difficult thing about putting this problem into a computer would be the inequalities. To take care of these, we introduce what we call slack variables. For each constraint except for positivity, introduce a new positive slack variable to turn the inequality into equality. I.e., $x_2 \leq 2$ becomes $x_2 + s_1 = 2$, $x_1 + x_2 \leq 3$ becomes $x_1 + x_2 + s_2 = 3$ and $2x_1 + x_2 \leq 5$ becomes $2x_1 + x_2 + s_3 = 5$. This changes our problem so that we now have five variables; $x_1, x_2, s_1, s_2, s_3 \geq 0$. This gives us a "larger" (in the sense that it is higher dimensional) feasible region. Now, if we consider the extreme points of the initial feasible region, we can uniquely determine values for each slack variable that will fulfill all of the constraints.

Extreme Points of initial feasible region	x_1	x_2	s_1	s_2	s_3
(0,0)	0	0	2	3	5
(0,2)	0	2	0	1	3
(1,2)	1	2	0	0	1
(2,1)	2	1	1	0	0
$(\frac{5}{2}, 0)$	$\frac{5}{2}$	0	2	$\frac{1}{2}$	0

Note, in this table in each case there are 2 variables equal to zero and each variable is 0 twice (can we generalize this?).

<u>Definition:</u> basic feasible solution of the linear programming problem

maximize $\quad f = c \cdot x$

subject to $\quad Ax = b$

$x \geq 0$

is a vector y satisfying: (1) at least $n - m$ of its components are 0, where n is the number of variables and m is the number of constraints, and (2) the system $Ax = S$ can be written equivalently so that the columns of A corresponding to the nonzero variables are linearly independent.

Now, we can reformulate the constraints on problem 2 as follows:

$$Ax = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$$

Here, note that a basic feasible solution would be $y = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 3 \\ 5 \end{bmatrix}$, since it has exactly 3 nonzero entries (which meets the definition, since here n=5 variables and m=3 constraints), and if we look at the last 3 columns of A (corresponding to the nonzero entries in our basic feasible solution), those are clearly linearly independent. For the same reasons, $y = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \\ 3 \end{bmatrix}$ is also a basic feasible solution.

End of Day 10, Beginning of Day 12

With this groundwork, we can now present a couple results concerning our linear programming problems.

Theorem 1 If the feasible region S of a linear programming problem is nonempty and there are m columns of A that are linearly independent then every extreme point of S is a basic feasible solution. Conversely, every basic feasible solution is an extreme point of S.

You may discuss how this theorem makes sense since there is an obvious 1-1 correspondence between the extreme points of S and the extreme points of the feasible region in the original problem.

Theorem 2 Assume a region S is nonempty and bounded. Then the maximum value of f is taken at an extreme point. If the maximum value occurs at several points $y_1, \ldots, y_k$, then it will occur at all convex combinations of those points.

Remarks:

1. The maximum value never occurs in the interior of the region, unless f is constant.
2. Any local maximum is also a global maximum. Meaning, if a particular x is optimal for f locally, then it will be the optimal solution.

Of course, it is not always so easy to simply look at a problem and tell where the maximum ought to occur. However, the simplex algorithm will provide us with a method for finding the solution analytically. We introduce this method by going though an example.

Let's return to the winery problem, recall the setup:

Maximize $f = \frac{5}{4}x_1 + \frac{3}{2}x_2 + 2x_3$

$$\text{subject to: } \begin{cases} 2x_1 + x_3 \leq 200 \\ 2x_2 + 2x_3 \leq 150 \\ 2x_1 + x_2 + \frac{3}{2}x_3 \leq 90 \\ 2x_1 + x_2 + 2x_3 \leq 250 \\ 0 \leq x_1, x_2, x_3 \end{cases}$$

For the sake of simplifying notation, we will introduce slack variables as before, but now we will call then $x_4, \ldots x_7$. Also, let's solve for each of the slack variables to get:

$$(*)\begin{cases} x_4 = 200 - 2x_1 - x_3 \\ x_5 = 150 - 2x_2 - 2x_3 \\ x_6 = 90 - 2x_1 - x_2 - \frac{3}{2}x_3 \\ x_7 = 250 - 2x_1 - x_2 - 2x_3 \end{cases}$$

At this point in the process, we have:

Stage	Nonbasic Variables	Basic Variables	f
1	x_1, x_2, x_3	x_4, x_5, x_6, x_7	$\frac{5}{4}x_1 + \frac{3}{2}x_2 + 2x_3$

We will continue filling in this table as we go along. At this point, we need to find a basic feasible solution, we simply choose the most obvious one, $y^T = [\ 0\ \ 0\ \ 0\ \ 200\ \ 150\ \ 90\ \ 250\ ]$. It is easy to verify that this is in fact a basic feasible solution. Here we should note that with this choice of y, we have $x_1 = x_2 = x_3 = 0$ so $f = 0$. We would like to increase this value. The idea is that we will move to another basic feasible solution along a path that will only increase the value of f.

Now, in our equation for f, all of the variables have positive coefficients, so increasing any one would increase the value of f. At this point we could choose any of the nonbasic variables to increase, we will choose x_2. To proceed, solve for x_2 in each equation in which it occurs in $(*)$. This gives us:

$$(**)\begin{cases} 2^{\text{nd}}\text{ equation}: & x_2 = \frac{150 - x_3 - x_5}{2} \\ 3^{\text{rd}}\text{ equation}: & x_2 = 90 - 2x_1 - \frac{3}{2}x_3 - x_6 \\ 4^{\text{th}}\text{ equation}: & x_2 = 250 - 2x_1 - 2x_3 - x_7 \end{cases}$$

Since all variables are non-negative, these equations give us more restrictions on x_2, from the second equation, we know that $x_2 \leq 75$, the third implies $x_2 \leq 90$ and the fourth gives us that $x_2 \leq 250$. The most restrictive is the second equation, so we will choose this equation to solve for x_2. At this point, if we substitute this expression for x_2 into f, we then get f depending on x_1, x_3, x_5. Making this substitution and continuing with the table above, we have:

Stage	Nonbasic Variables	Basic Variables	f
1	x_1, x_2, x_3	x_4, x_5, x_6, x_7	$\frac{5}{4}x_1 + \frac{3}{2}x_2 + 2x_3$
2	x_1, x_3, x_5	x_2, x_4, x_6, x_7	$\frac{225}{2} + \frac{5}{4}x_1 + \frac{5}{4}x_3 - \frac{3}{4}x_5$

Additionally, $(*)$ becomes: $\begin{cases} x_4 = 200 - 2x_1 - x_3 \\ x_5 = 75 - \frac{x_3}{2} - \frac{x_5}{2} \\ x_6 = 15 - 2x_1 - x_3 - \frac{x_5}{2} \\ x_7 = 175 - \frac{3}{2}x_3 + \frac{x_5}{2} \end{cases}$

End of Day 12, Beginning of Day 13

We now have the option to increase either x_1 or x_3 in order to increase the value of f. We will choose to introduce x_3 as a new basic variable. In order to maintain the positivity of the old basic variables, the following conditions must be satisfied:

$$(***)\begin{cases} 1^{\text{st}}\text{ equation}: & x_3 \leq 200 \\ 2^{\text{nd}}\text{ equation}: & x_3 \leq \frac{75}{\frac{1}{2}} = 150 \\ 3^{\text{rd}}\text{ equation}: & x_3 \leq 15 \\ 4^{\text{th}}\text{ equation}: & x_3 \leq \frac{175}{\frac{3}{2}} = \frac{350}{3} \end{cases}$$

In this instance, the third equation is the most restrictive, so that is the one we will use to solve for x_3. From this equation we get $x_3 = 15 - 2x_1 + \frac{x_5}{2} - x_6$. Now, if we substitute this into the previous equation for f we would obtain f written in terms of x_1, x_5, and x_6, with this in mind we can solve for all of the other new basic variables:

$$\begin{aligned} x_2 &= \frac{135}{2} + x_1 - \frac{3}{4}x_5 + \frac{x_6}{2} \\ x_4 &= 185 - \frac{x_5}{2} + x_6 \\ x_7 &= \frac{305}{2} + x_1 + \frac{x_5}{4} + \frac{3}{2}x_6 \end{aligned}$$

Summarizing where we are in this step, we see:

Stage	Nonbasic Variables	Basic Variables	f
1	x_1, x_2, x_3	x_4, x_5, x_6, x_7	$\frac{5}{4}x_1 + \frac{3}{2}x_2 + 2x_3$
2	x_1, x_3, x_5	x_2, x_4, x_6, x_7	$\frac{225}{2} + \frac{5}{4}x_1 + \frac{5}{4}x_3 - \frac{3}{4}x_5$
3	x_1, x_5, x_6	x_2, x_3, x_4, x_7	$\frac{525}{4} - \frac{5}{4}x_1 - \frac{1}{8}x_5 - \frac{5}{4}x_6$

At this point, all of the coefficients on all of the nonbasic variables in our objective function are negative. Therefore, f will attain a local (and therefore global) maximum when $x_1 = x_5 = x_6 = 0$. Since we have equations for all of the basic variables in terms of the nonbasic variables, we can easily see this implies $x_2 = \frac{135}{2}$, $x_3 = 15$, $x_4 = 185$, $x_7 = \frac{305}{2}$.

Recall, the only variables from the original statement of the problem (i.e. the variables which are not slack variables) was x_1, x_2, and x_3. So the optimal solution will be attained when these three variables take on the values listed above, and the value of f at this point will be $\frac{525}{4}$.

5 Day 19-20: Integer Programming and Graph Theory

Recall, with an LP problem, our goal was either to maximize or minimize an objective function subject to certain linear constraints. An integer program is the same, though we simply add the restriction that all variables must be integers. For certain problems (including the one we are considering in this class), having discreet variables is much more realistic than continuous ones. To begin with a discussion of a discreet function, we consider the Boolean Function:

$f(x_1, x_2, x_3) = (x_1 \vee \bar{x_2})\&(x_1 \vee x_3)\&(\bar{x_2} \vee \bar{x_3})$

Here, we must clarify notation. The symbol "$\vee$" is read "or," "$\bar{x_1}$" is read " not x_1," and "&" is simply "and." Now, to begin we consider that each of the three variables in our function takes on one of two values, either "T" or "F" (can be thought of as true or false). To see how to evaluate a point with various different specific values for each of the variables, consider $f(T,T,F)$. The output of f contains three separate statements (note: it is not necessary for the number of statements to be equal to the number of variables). Consider the first statement, $(x_1 \vee \bar{x_2})$, read "x_1 or not x_2." In order for this statement to be satisfied, we must have either x_1 being true or x_2 being false, or both. Notice, since $x_1 = T$ in this case, we have this first statement is true. We can similarly evaluate each of the other two statements, we will conclude that each of the other statements is true. Therefore, we can write: $f(T,T,F) = T \& T \& T = T$.

In order for a function to evaluate to T, all three of the statements must evaluate to T, if even one statement evaluates to F, we would have the function evaluate to F. For this reason, (T,T,F) is called a satisfying assignment.

In this way, we can check individual points to figure out how they evaluate when substituted into f. Although, this is a very inefficient way to find satisfying assignment. In general it is better to formulate this as an integer program. Since each variable can only take on one of two values, we add the constraint that $x_1, x_2, x_3 \in \{0,1\}$, where we think of a value of 0 corresponding to F and a value of 1 corresponding to T. Now we can translate the three statements in f to three separate constraints. For the first statement, $(x_1 \vee \bar{x_2})$, we add the constraint $x_1 + (1 - x_2) \geq 1$. Notice this constraint will be satisfied if and only if $x_1 = 1$ or $x_2 = 0$, or both. Similarly, the other two statements translate to constraints $x_1 + x_3 \geq 1$ and $(1 - x_2) + (1 - x_3) \geq 1$.

For another example, translate the problem $g(x_1, x_2) = (x_1 \vee x_2) \& (\bar{x_1} \vee x_2) \& (x_1 \vee \bar{x_2}) \& (\bar{x_1} \vee \bar{x_2})$ into an integer program. To do this, we simply specify:

$$\begin{cases} x_1 + x_2 \geq 1 \\ (1 - x_1) + x_2 \geq 1 \\ x_1 + (1 - x_2) \geq 1 \\ (1 - x_1) + (1 - x_2) \geq 1 \\ x_1, x_2 \in \{0,1\} \end{cases}$$

It is an easy exercise to graph all of these constraints and see that the only point in the feasible region is $\left(\frac{1}{2}, \frac{1}{2}\right)$. Therefore, there is no solution to the integer program, and therefore no satisfying assignment for g. However, if we relax the constraint that $x_i \in \mathbb{Z}$ and allow $x_i \in \mathbb{R}$, there will be a solution. We refer to this as the LP relaxation of the integer program.

To begin with basic properties of integer programs, suppose we have an integer program

$$\begin{cases} \text{maximize}\, c^T x \\ \text{subject to}\, Ax = b \\ \quad x \geq 0 \\ \quad x_i \in \mathbb{Z} \end{cases}$$

Suppose solving this program gives us an optimal value for the objective function of $\hat{z}$. If we then consider the LP relaxation:

$$\begin{cases} \text{maximize}\, c^T x \\ \text{subject to}\, Ax = b \\ x \geq 0 \end{cases}$$

and obtain an optimal value of z^*, then we can automatically conclude $\hat{z} \leq z^*$. This is true since the feasible region for the integer program is contained in the feasible region for the LP relaxation, every possibility for a solution to the integer program is also a possibility for the LP relaxation. Keeping this in mind, we present two possible methods for solving an integer program.

1. Branch and Bound: The idea for this method is to divide up the feasible region of the LP relaxation, looking for the integral parts.

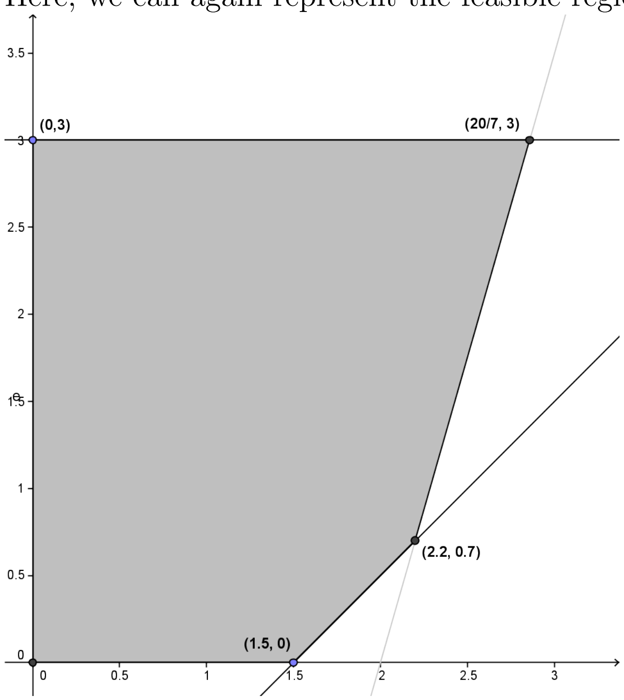
2. Cutting Planes: Here, we add inequalities to the LP relaxation that reduces the size of the feasible region, but keeps all optimal integral solutions.

Now, we will present the method of branch and bound with an example.

Consider the problem: Max $4x_1 - x_2$ subject to:

$$\begin{cases} 7x_1 - 2x_2 \leq 14 \\ x_2 \leq 3 \\ 2x_1 - 2x_2 \leq 3 \\ x_1, x_2 \geq 0, \quad x_1, x_2 \in \mathbb{Z} \end{cases}$$

Here, we can again represent the feasible region graphically:



If the solve the LP relaxation, we obtain a maximum value of the objective function $z^* = \frac{59}{7}$ attained at $\left(\frac{20}{7}, 3\right)$. The idea for branch and bound is basically that the LP relaxation ought to give us a good estimate for the IP. Since x_1 is not integral in this case, we "branch" off of this variable.

Consider two new regions, one where $x_1 \geq \lceil \frac{20}{7} \rceil = 3$ and one where $x_1 \leq \lfloor \frac{20}{7} \rfloor = 2$. In other words, we consider two new LP problems, one with the first constraint and the other with the second.

End of Day 19, Beginning of Day 20

If we consider the first branch, we can easily see that there is actually no feasible region, so we disregard this branch (there is no solution). However, consider the second branch, $x_1 \leq 2$. If we solve the LP problem with this extra constraint, we obtain a maximal value of the objective function of $\frac{15}{2}$ obtained at $x_1 = 2$, $x_2 = \frac{1}{2}$. Note $\frac{15}{2} < \frac{59}{7}$, so as we decrease our feasible region the max value of the objective function is non-increasing.

Now, this solution is still not integral. Therefore, we can branch again, off of x_2 this time. We will keep the same constraints as before (including $x_1 \leq 2$, since we are still on that branch), but add either the constraint $x_2 \leq \lfloor \frac{1}{2} \rfloor = 0$ or $x_2 \geq \lceil \frac{1}{2} \rceil = 1$.

Considering the second branch first, we summarize the LP relaxation problem, maximize $4x_1 - x_2$ subject to:

$$\begin{cases} 7x_1 - 2x_2 \leq 14 \\ x_2 \leq 3 \\ 2x_1 - 2x_2 \leq 3 \\ x_1 \leq 2 \\ x_2 \geq 1 \\ x_1, x_2 \geq 0 \end{cases}$$

Solving this LP relaxation yields maximum value of 7 obtained at $(2, 1)$. Now we have found an integral solution, but we still are not certain whether it is optimal. There is still one more branch we have not considered, $x_2 \leq 0$. Solving this LP relaxation yields an objective function value of 6 obtained at $x_1 = \frac{3}{2}$, $x_2 = 0$. At this point, we could branch again off of x_1, but any value we obtain by restricting the feasible region further would obtain maximum values less than or equal to 6. Since we have already found an integral solution with a value of 7 we stop with this branch.

Now, we have considered every possible branch, so we can conclude that our solution to the integer program is obtained at $(2, 1)$ with a value of 7. To summarize the branch and bound method for solving integer programs:

1. Build a search tree (for a maximization problem)

2. For each node of the search tree, first solve the LP relaxation.

3. If an integral solution is obtained, compare it to existing bounds on the solution we know. I.e. each integral solution gives a lower bound on what the solution to the integer program will be.

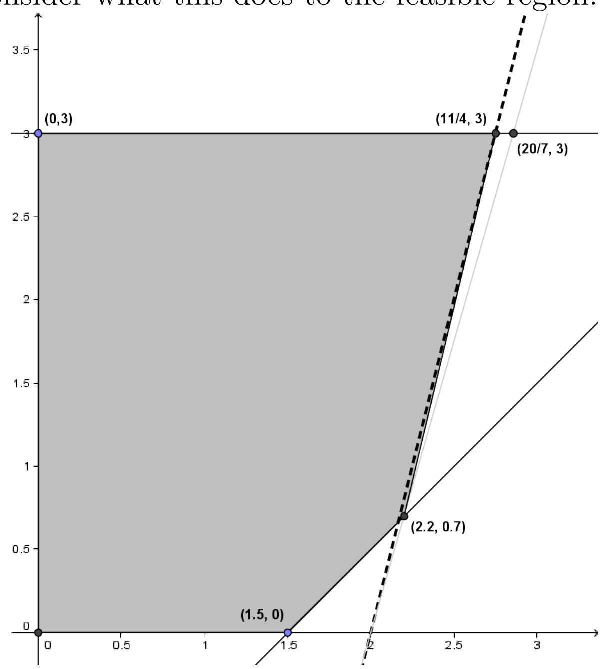
4. If the maximal solution on a node is less than any existing lower bound, stop with this node. Otherwise, divide the feasible region again by branching off of any fractional variable.

The other technique we mentioned for solving an integer program is the method of cutting planes. Recall the idea was to add inequalities to reduce the feasible region while preserving the integral solutions. To illustrate how we may do this, go back to the previous example. In the original LP relaxation we obtained a maximum value of the objective function of $\frac{59}{7}$ at $x_1 = \frac{20}{7}$ and $x_2 = 3$.

Now, recall the objective function is $4x_1 - x_2$, where the coefficients are both integers. Therefore, substituting integer values in for x_1 and x_2 will yield an integer value for the objective function. This leads us to suspect that we ought to add a constraint of the form:

$$4x_1 - x_2 \leq \left\lfloor \frac{59}{7} \right\rfloor = 8 \text{ or } 4x_1 - x_2 \geq \left\lceil \frac{59}{7} \right\rceil = 9$$

Notice that the second constraint is not possible. Therefore, we could add the first constraint. Consider what this does to the feasible region:



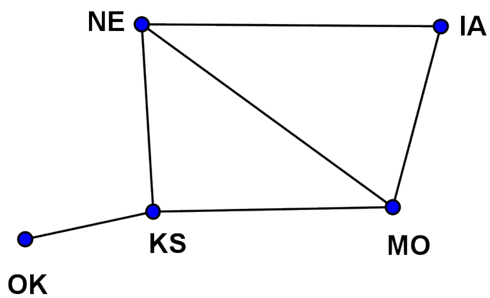
Notice how no integral solutions are cut out when this constraint is added, but it does reduce the size of the feasible region. Therefore, in this case this would seem to help us get closer to the solution. However, knowing which cutting planes to add in general is difficult. In fact, there is no single way to determine exactly which cutting planes ought to be added, and any cutting plane technique is problem specific.

Graph Theory

Here we include a very brief introduction to graph theory.

Definition: A *graph* G is a collection of vertices, call it $V(G)$, and a set of ordered pairs of vertices, which we call the edges.

We can represent a graph visually, simply drawing the vertices in the plane and drawing an line segment between them for each time the two of them are the members of an ordered pair in $V(G)$. For example, if we choose our collection of vertices to be the states Nebraska, Iowa, Kansas, Missouri, and Oklahoma, and choose to include an ordered pair for each set of states that border each other, it would yield a graph that looks like the following:



Now, a couple more definitions:

Definition: For x and y, two vertices of a graph, the following statements are equivalent:

a. $x \leftrightarrow y$

b. x is adjacent to y

c. (x, y) is an edge

d. x is a neighbor of y

Definition: An *independent set* is a set of vertices, no two of which are adjacent.

One typical problem we can ask is how to find the largest independent set. This may arise in practice, perhaps at a university we would like to survey a population of students, no two of which are enrolled in the same course. We may actually model this situation as an integer program.

In general, to formulate an integer program there are three things we need: 1. Variables, 2. An objective function, and 3. Constraints. The variables are always a good place to start. In this case it makes sense to define $x_i \in \{0, 1\}$, where i run from 1 to the population of students in the university. Define $x_i = 1 \Leftrightarrow$ student i is in our chosen independent set.

Now, since we want to choose the largest independent set we can, we have the objective function:

$$\text{Max} \sum_i x_i$$

Clearly, this simply counts the number of students in the set. Now, there are a couple different ways we could approach the constraints. The most direct way (going straight from the way the problem was stated) is to look at each individual edge. We could simply say for each $(i, j) \in V(G)$, $x_i + x_j \leq 1$.

However, this would result in a very large number of constraints. One more efficient way to get the same result, instead of looking at each individual student, we could look at each class as a whole. If we let C be an index set which indicates every student enrolled in one particular class, then for each such C we would add the constraint $\sum_{i \in C} x_i \leq 1$. This results in far fewer constraints, and may be more practical to implement.

6 Day 23: The Traveling Salesman Problem

For the recycling projects we dealt with this semester, it was necessary to incorporate routing into the models. Rather than continuing with the assumption that each truck will go to an individual location and then return to a central location after each stop is not reflective of what actually happens. Eventually, it was necessary to come up with ideas about how to divide the various sites into multiple tours for the different days of the week. However, we began with finding an optimal tour going to all of the sites in one day. See the individual reports for a more detailed explanation.

Statement of Problem

The Traveling Salesman Problem (TSP) is a well-studied problem involving routing. Suppose that a salesman wishes to visit each of n cities and return to his starting city at the end of the trip. The salesman wishes to minimize the total distance (or total cost) of his total trip. We assume that the salesman can go from any city directly to any other city, and hence he wishes to visit each city exactly once. We call such a complete trip a *tour*, and the tour that minimizes the objective function an *optimal tour*.

Let the cities be numbered $1, 2, \ldots, n$. We may think of the salesman starting and ending at city 1, though since a tour is cyclic, any city can be chosen as the starting point.

One method of solving a TSP to use brute-force searching: try every possible tour to find one with minimum cost. The number of tours is $(n-1)!$ since each permutation of the $2, \ldots, n$ describes the order in which cities are visited when starting from city 1, making this method unpractical even for small values of n (such as $n = 20$).

In 1954, George Dantzig, Ray Fulkerson, and Selmer Johnson of the RAND Corporation first formulated this combinatorial optimization problem as an integer programming (IP) problem. On the face of it, IP seems to be a strange tool to use for TSP, but in practice this turns out to be one of the most efficient means of solving TSPs.

Integer Programming Formulation

To formulate an IP, we need to decide what our variables, constraints, and objective function should be. The key in choosing the variables is to consider the objective function: we wish to minimize the total cost of the intercity trips that the salesman takes. Thus, we use binary variables x_{ij}, for $1 \leq i \leq n$, $1 \leq j \leq n$, and $i \neq j$, where

$$x_{ij} = \begin{cases} 1, & \text{if the salesman takes the trip from city } i \text{ to city } j, \\ 0, & \text{otherwise.} \end{cases}$$

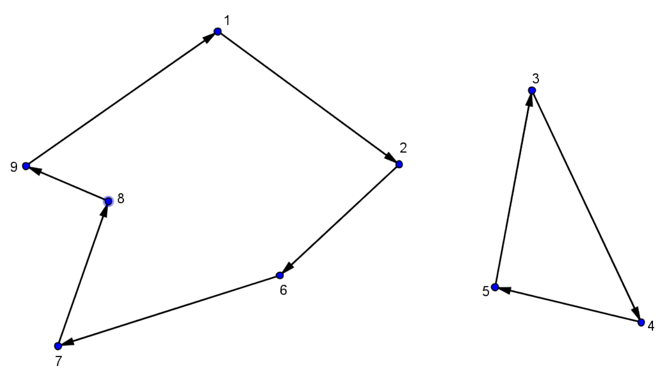
If the cost of the trip from city i to city j is c_{ij}, then the objective function is

$$\min \sum_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n \\ i \neq j}} c_{ij} x_{ij}.$$

We now wish to add constraints to guarantee that we have a valid tour. For each city i, exactly one outgoing trip is taken, and exactly one incoming trip is taken. Thus, we have the following constraints:

$$\forall 1 \leq i \leq n, \quad \sum_{\substack{j=1 \\ j \neq i}}^{n} x_{ij} = 1 \quad \text{(each city has one outgoing edge)}$$
$$\forall 1 \leq i \leq n, \quad \sum_{\substack{j=1 \\ j \neq i}}^{n} x_{ji} = 1 \quad \text{(each city has one incoming edge).}$$

However, these constraints are not sufficient, as a feasible solution might contain disconnected *subtours* instead of one connected tour that visits every city. For example, we want to avoid getting answers which look like the following:



We thus need to add *subtour elimination constraints*. For any nonempty proper subset S of $\{1, \ldots, n\}$, let $\overline{S}$ denote its complement; that is, $\overline{S} = \{1, \ldots, n\} \setminus S$. To include all cities, a valid tour must contain a trip from a city in S to a city in $\overline{S}$; perhaps there are several such trips. Thus, we add the constraints

$$\forall S \subseteq \{1, \ldots, n\},\ S \neq \emptyset,\ S \neq \{1, \ldots, n\}, \quad \sum_{\substack{i \in S \\ j \in \overline{S}}} x_{ij} \geq 1.$$

Notice that there are an exponential number of constraints: $2^n - 2$ (actually, we don't need the constraints when $|S| = 1$ or $|S| = n - 1$, since these are implied by the incoming and outgoing edge constraints from above). It is not practical to construct, much less solve, an IP with all of these constraints.

We can, however, use the constraints in an iterative process similar to the way cutting planes are used. We first solve the IP without any subtour elimination constraints. If the solution obtained is one connected tour, then we have an optimal solution. Otherwise, we add a subtour constraint for an appropriate S; for example, S can be chosen to be set of cities in the subtour containing city 1. We then resolve the IP with this new constraint and repeat.

The process stops when the solution obtained is one connected tour. We are guaranteed to obtain an optimal solution, since all subtours are eliminated. Though many subtour elimination constraints may be needed, in practice only a small number are needed.

Alternate Subtour Elimination Constraints

An alternate set of subtour elimination constraints can be formulated through the addition of new variables u_i for $2 \leq i \leq n$. The variables u_i do not have lower or upper limits, and are not constrained to be integral. We add the constraints

$$\forall 2 \leq i \leq n,\ 2 \leq j \leq n,\ i \neq j, \quad u_i - u_j + (n-1)x_{ij} \leq n - 2.$$

Since there are only a quadratic number of constraints, all the constraints can be added initially and the IP solved once.

Why do these alternate constraints eliminate subtours while preserving all connected tours?

Claim 1: The alternate constraints eliminate subtours.

Proof. Suppose that we have a solution where $i_1, \ldots, i_k$ are the cities (in order) of a subtour that avoids city 1. Then $x_{i_1 i_2}, x_{i_2 i_3}, \ldots, x_{i_{k-1} i_k}, x_{i_k i_1}$ are all 1. We consider the corresponding constraints:

$$\begin{aligned}
u_{i_1} - u_{i_2} + (n-1)x_{i_1 i_2} &\leq n-2 \\
u_{i_2} - u_{i_3} + (n-1)x_{i_2 i_3} &\leq n-2 \\
&\vdots \\
u_{i_k} - u_{i_1} + (n-1)x_{i_k i_1} &\leq n-2
\end{aligned}$$

S umming all the constraints, we have cancellation on the left side, and so obtain

$$k(n-1) \leq k(n-2),$$

which is a contradiction since k is positive. □

Claim 2: The alternate constraints do not eliminate any connected tours.

Proof. Suppose that $1, i_2, i_3, \ldots, i_n$ are the cities (in order) of a tour through every city. Then we can construct a feasible solution. The variables x_{ij} are set to 1 if and only if ij is an edge along the tour. We set $u_{i_t} = t$; that is, u_j is t if j is the tth city along the tour. If ij is not an edge along the tour, then $u_i - u_j \leq n-2$, and hence the constraint is satisfied. If ij is an edge along the tour, then $u_i - u_j = -1$, so $u_i - u_j + (n-1)x_{ij} \leq n-2$, and again the constraint is satisfied. The previous method of adding constraints along the tour does not produce a contradiction here, since city 1 is not included in the constraints. □

Note that although we interpret the variables u_i as positions along the tour, we do not constrain the variables to be integral or nonnegative. This is purely for efficiency reasons, as the additional constraints require more time and space by an IP solver.

7 Appendix

Homework 1

The projects this semester will revolve around recycling several commodities and analyze the benefits of recycling vs. the costs involved in recycling. You are to research and write about one of these items. Your essay (of about 3-4 pages) should contain information about the specific material/item such as:

1. general information: material characteristics such as density, varieties, compactibility, etc.
2. information on amounts recycled every year and percentage of recycled material vs. new material produced;
3. costs, energy, and resources of producing new material; availability of the raw material;
4. costs, energy, and resources of recycling the material;
5. benefits and disadvantages of recycling the material for humans and the environment; such as minimizing gas emissions from landfills, etc.
6. ease of recycling etc.

Please include all references that you consulted, including websites, friends, movies etc.

1. plastic
2. newspaper
3. mixed paper
4. cardboard
5. aluminum
6. steel food cans
7. glass
8. costs of transportation I (gas)
9. costs of transportation II (labor, wear on the road, costs of using trucks)
10. recycling nationwide (rates of participation, changes in rates of participation over time, factors and policies that contribute to the change in participation)
11. The EPA lists the first benefit of recycling to be: “Recycling protects and expands U.S. manufacturing jobs and increases U.S. competitiveness”. Write an essay that will quantify this statement and will provide proof for the ideas contained in it.

Homework 2

Your solutions should use proper English and complete statements. Make sure your notation and assumptions are stated clearly at the beginning of each problem. **Please read your solutions entirely before you turn them in.**

1. **Guesstimates**. (20 points) Provide an order of magnitude in the following problems. Clearly write your **assumptions** and **arguments**.
   (a) How many dentists are in Las Vegas?
   (b) How many pairs of shoes can be made from a cow? (consider a spherical cow and a spherical shoe.)
   (c) How much money does an individual spend during his/her lifetime? Include here the expenses made by the individual's parents during his/her childhood. Please state your assumptions regarding income, inflation, interest rates etc.

2. **Natural Growth Models** (30 points)
   (a) Use the natural growth model to predict the US population if the growth rate is assumed to be the same as the growth rate of a) 1980; b) 1990; c)2000.
   (b) Assume an investment grows according to the natural growth model at a rate of 20% a year. How long will it take to double its value? Explain why this time does not depend on the amount of money initially invested in the account. Next assume that at the beginning of every year, starting with the second year, $100 is taken out of the account. Under the new circumstances how long will it take for the investment to double its value? Does your answer depend now on the initial investment? Why or why not?

3. **Mathematical Modeling**. (30 points) Write an essay about a mathematical model (one you may have encountered in a Calculus, Differential Equations, Linear Programming, Probability, or other mathematics course). Your write up should contain the following:
   (a) general description of the model, where and how it is used.
   (b) mathematical set up; all mathematical quantities, notation, range of parameters are specified.
   (c) the equations describing the state/phenomena.
   (d) mathematical conclusions for the model (if it can be solved, include the solution); results, possible graphs.
   (e) physical interpretation of the results.

Homework 3

This is an individual assignment. You may only consult the instructor with any questions that you may have. In order to receive full credit, clearly present and motivate all the steps in your solutions.

1. (25 points) A hospital patient is required to have at least 90 units of drug I and 120 units of drug II. The drugs are both contained in two substances S_1 and S_2. Suppose a gram of S_1 contains 6 units of drug I and 4 units of drug II, and a gram of S_2 contains 3 units of drug I and 3 units of drug II. But in addition, each gram of S_1 contains 2 units of a mildly toxic drug and each gram of S_2 contains 1 unit of this other undesirable drug. How much of each substance should be given to the patient to achieve the medication requirements with minimal dosage of the toxin? How much of the toxin does the patient receive with this optimal mixture? Show the steps of your work.

2. (25 points) Solve the problem below, showing the steps of your work. Show that the feasible region is unbounded.

$$\text{maximize:} \quad f = -2x_1 + x_2 + x_3$$

$$\text{subject to:} \quad \begin{cases} -x_1 + x_2 + x_3 \leq 2 \\ x_1 - x_2 + x_3 \leq 2 \\ x_1, x_2 \geq 0. \end{cases}$$

3. (40 points) *Linear best fit to data.* Consider a set of n data points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ that are the observed outcomes of an experiment. If we think that y_i is linearly related to x_i, we would like to find the *linear best fit*; that is, the constants m and b such that $y = mx + b$ is the equation of the line that best fits that data.

   (a) How do we measure the "fit" of the data? One way is to use the ℓ_1-norm, which is the sum of the absolute values of the differences between the observed value y_i and the predicted value $mx_i + b$:

   $$\sum_{i=1}^{n} \left|(mx_i + b) - y_i\right|.$$

   We wish to minimize this function over all values of $m, b \in \mathbb{R}$. Formulate an equivalent linear program that solves this minimization problem. (*Hint.* Introduce a new variable to eliminate each absolute value.)

(b) Find the linear best fit under the ℓ_1-norm for the data: (5.38, 8.47), (0.53, -0.17), (8.8, 13.05), (7.77, 14.14), (0.55, 2.63), (6.14, 7.43), (10.0, 14.5), (2.4, 4.92), (9.03, 10.41), (6.84, 9.85), (1.23, 3.84), (0.17, 3.61), (3.68, 3.69), (8.27, 12.15), (3.02, 7.34), (4.85, 3.63), (6.26, 5.39), (9.94, 16.37), (3.03, 3.78), (5.41, 8.41).

You may use computer software for the calculations.

(c) *Least-squares fitting* minimizes the ℓ_2-norm instead, which is the sum of the squares of the differences between the observed value y_i and the predicted value $mx_i + b$:

$$\sum_{i=1}^{n} \left[(mx_i + b) - y_i\right]^2 .$$

Using either inner products and vector spaces, or calculus, one can obtain the following formula for the least squares fit. Set

$$S_x = \sum_{i=1}^{n} x_i, \quad S_y = \sum_{i=1}^{n} y_i, \quad S_{xx} = \sum_{i=1}^{n} x_i^2, \quad S_{xy} = \sum_{i=1}^{n} x_i y_i.$$

Then the estimates of m and b are

$$\hat{m} = \frac{nS_{xy} - S_x S_y}{nS_{xx} - S_x^2}, \qquad \hat{b} = \frac{S_y}{n} - \hat{m}\frac{S_x}{n}.$$

(d) Choose one of the data points and move its location up or down vertically. For each of your choices, plot (on the same axes) these data points and the two lines obtained from fitting using the ℓ_1 and ℓ_2 norms; observe what happens to the best fit line with respect to the two norms as you slide the point up and down. Are the two lines always the same? When might you want to use one or the other?

Homework 4

This is an individual assignment. You may only consult the instructor with any questions that you may have. In order to receive full credit, clearly present and motivate all the steps in your solutions.

1. Solve the following integer program using the technique of branch-and-bound. Show the steps of your solution.

$$\begin{array}{rrrcl} \min & x_1 & +x_2 & & \\ \text{subject to} & x_1 & +2x_2 & \geq & 2 \\ & -x_1 & +4x_2 & \leq & 3 \\ & x_1, & x_2 & \geq & 0, \text{ integer} \end{array}$$

2. Suppose that you have 10 objects that weigh 7, 9, 17, 72, 74, 75, 77, 86, 89, 95 ounces, respectively. You have a knapsack that can hold at most 472 ounces. What is the maximum weight of the above objects that you can carry in your knapsack? Note that there is only one of each object.

   (a) Formulate an integer program that models this problem, and solve the integer program (using any means).

   (b) Compare the solution to the integer program to the solution of its linear programming relaxation. Can you explain this? Make a conjecture about the solution to the linear programming relaxation for an arbitrary knapsack problem.

3. Pick ten cities or towns within one state and solve the Traveling Salesman Problem on these cities. To compute the distances between the cities, determine the latitude and longitude of each city, and then use the Pythagorean theorem.