Numerical Range of a Composition Operator on the Hardy Space

Julia Balukonis[1], Laney Bowden[2], Fatme Hourani[3], and Ellie Lochner[4]
Advisor: John Clifford
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Abstract

For a bounded operator T on a Hilbert Space H, the numerical range of T is the subset $W(T)$ of C given by

$$W(T) = \{< Tx, x > \ : \ \|x\| = 1\}.$$

We study the numerical range of the composition operator, C_A, on the Hardy space $H^2(\mathsf{B}_n)$ where A is an $n \times n$ matrix that is a self-map of the unit ball B_n. We show $W(A)$ $W(C_A)$. In the special case where A is a weighted shift, $W(C_A) =$ convex hull$(W(A) \quad \{1\})$. We completely characterize the numerical range of the operator when A is unitarily similar to a Jordan-normal form that maps the ball to the ball by decomposing our operator into the direct sum of shifts and normal operators.

Introduction

The Hardy space on the open unit ball denoted $H^2(\mathsf{B}_n)$ is the set of holomorphic functions f on B_n such that

$$\sup_{0<r<1} \int_{\mathsf{S}_n} |f(r\zeta)|^2 d\sigma(\zeta) < \infty.$$

The Hardy space can be further characterized as a Hilbert space with inner product

$$f, g \ = \sup_{0<r<1} \int_{\mathsf{S}_n} f(r\zeta)\overline{g(r\zeta)} d\sigma(\zeta)$$

This can then be decomposed into a direct sum of orthogonal subspaces of homogeneous polynomials, that is, $H^2(\mathsf{B}_n) = \quad \overset{\infty}{_{k=0}} H_k$ where,

$$H_k^n = \text{span}\{z_n^\alpha \ : \ |\alpha| = k\}.$$

Our composition operator is defined as

$$C_A f(z) = (f \circ A)(z) = f(Az),$$

where $f \quad H^2(\mathsf{B}_n)$ and $A : \mathsf{B}_n \to \mathsf{B}_n$. Note, $C_A : H_k^n \to H_k^n$; that is, H_k^n is an invariant subspace of C_A.

Backward Shift

An $n \times n$ backwards shift on C^n can be represented by

$$T_n = \begin{pmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

The composition operator induced by the shift is

$$C_{T_n} f(z) = f \circ (T_n z) = f(z_2, \dots, z_n, 0).$$

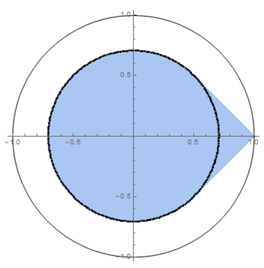
Numerical Range of the Shift

A matrix representation, M_k, of C_{T_n} restricted to the finite dimensional subspace H_k is

$$\begin{pmatrix} (T_n) & & & & & \\ & \begin{pmatrix} (T_{n-1}) & & \\ & \dots & \\ & & (T_{n-1}) \end{pmatrix} & & & & \\ & & \dots & & & \\ & & & \begin{pmatrix} (T_1) & & \\ & \dots & \\ & & (T_1) \end{pmatrix} \end{pmatrix}$$

$$= T_n \quad T_{n-1} \quad \cdots \quad T_{n-1} \quad \cdots T_1 \quad \cdots \quad T_1.$$

It is well known that the numerical range of the backward shift is the closed disc of radius $\cos((n+1)/\pi)$. Thus $W(M_k) = W(T_n) = \cos((n+1)/\pi)\overline{\mathsf{D}}$



$W(C_{T_n})$

Theorem 1 (BBCHL-2018)

If A is a weighted shift on C^n where $A \quad \leq 1$ then $W(C_A) =$ convex hull $(W(A) \quad \{1\})$. In particlar if A is the backward shift then

$$W(C_A) = \text{convex hull}\left(\{z : |z| \leq \cos\left(\frac{n+1}{\pi}\right)\} \quad \{1\}\right)$$

Numerical Range of Normal Composition Operators

It is well known that C_A is normal if and only if A is a normal n by n matrix and $A \quad \leq 1$. A normal matrix A is unitarily diagonalizable, that is there exists a unitary matrix, U such that $U \ AU = D$ where D is a diagonal matrix with the eigenvalues of A along the diagonal. The numerical range of a normal matrix is the convex hull of its eigenvalues. Let $\{\lambda_1 \cdots \lambda_m\}$ be the eigenvalues of A. Then

$$W(A) = \text{convex hull } (\{\lambda_1 \cdots \lambda_m\}).$$

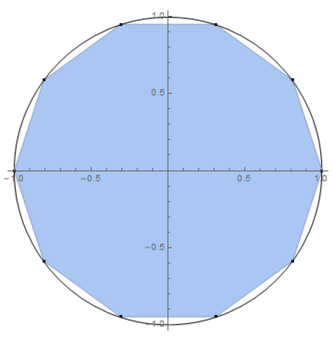
Similarly, $W(C_A) = W(C_D)$. To find $W(C_D)$, let $\{\lambda_1, \cdots, \lambda_m\}$ be the eigenvalues of A and $P = \{\lambda_1^{\alpha_1} \cdots \lambda_m^{\alpha_m}\}$, all possible products of eigenvalues. If C_A is normal,

$$W(C_A) = \text{convex hull } (P)$$

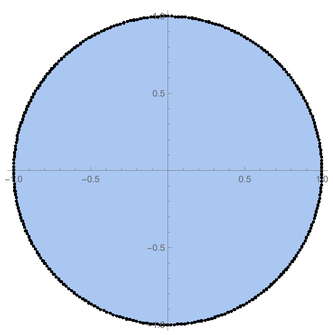
Theorem 2 (BBCHL-2018)

Suppose C_A is a normal operator.

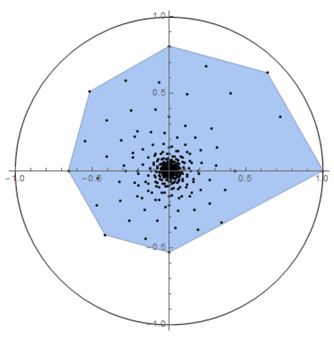
1. If the eigenvalues of A are of the form $\lambda = e^{i\frac{p}{q}\pi}$ where $p, q \quad \mathsf{Z}$, then $W(C_A)$ is a regular polygon inscribed in the unit circle
2. If at least one eigenvalue of A is of the form $\lambda = e^{i\theta\pi}$ where $\theta \quad \mathsf{Q}^{\{}$, then $W(C_A) = \mathsf{D} \quad P$ where $P = \{\lambda_1^{\alpha_1} \cdots \lambda_m^{\alpha_m}\}$, a dense set on the unit circle.
3. Otherwise, $W(C_A)$ is a polygon.



Regular Polygon



D P



Polygon

Main result

If A is unitarily similar to a Jordan normal block diagonal matrix and A is a self-map of the unit ball then the matrix representation of C_A restricted to the subspace H_k^n is

$$M_k = T_i \quad \cdots \quad T_j \quad D_k.$$

Using this observation we can reduce the proof of the Main Theorem to Theorem 1 and Theorem 2.

Main Theorem (BBCHL-2018)

If A is unitarily similar to a Jordan normal block diagonal matrix then the $W(C_A)$ is equal to the convex hull of $W(T_i) \quad W(D) \quad \{1\}$.

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Contact Information

- [1]Providence College, jbalukon@friars.providence.edu
- [2]Colorado State University, lbowden1@rams.colostate.edu
- [3]University of Michigan - Dearborn, fhourani@umich.edu
- [4]University of Wisconsin - Eau Claire, lochneen0760@uwec.edu