Cops and Robbers on Toroidal Chess Graphs

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Introduction to Cops and Robbers

Played on graphs, Cops and Robbers is a pursuit-evasion game between a set of cops and a robber, and is played with the following rules:

1. Given any mathematical graph G, k cops choose up to k vertices of G as their starting positions (multiple cops may occupy a single vertex), and then the robber chooses his vertex.
2. The game proceeds with the cops and robber alternating turns by each choosing to stay on their current vertex or moving to an allowable vertex*.
3. The cops win if, in a finite number of moves, at least one cop can occupy the same vertex as the robber.
4. The robber wins if he can guarantee to never have a cop occupy the same vertex as he occupies.

*Allowable vertices are those that a cop or robber can move to based on their movement assignment.

Cops and Robbers on a Torus

Assume G is an $n \times n$ chess graph on the torus. A *chess graph* is an $n \times n$ array of vertices, with an edge between all horizontal "neighboring" vertices as well as all vertical "neighboring" vertices.

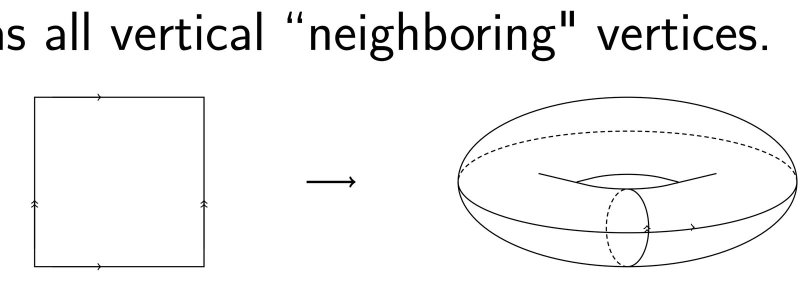

Figure 1: Identifying edges of a square to create a torus

Types of Moves

- **On Foot**: allowable vertices are those adjacent to the occupied vertex.
- **Knight**: allowable vertices are those located 2 columns and 1 row, or, 1 column and 2 rows away from the occupied vertex.
- **Chief**: allowable vertices are those located in the same row, column, or diagonal as the occupied vertex.
- m**-speedy**: allowable vertices are those in the same row or column located up to a distance of m from the occupied vertex occupied.

Definition

The **cop number** of a graph G, denoted $c(G)$, is the minimum number of cops required to guarantee the existence of a winning strategy for the cops, regardless of the robber's initial position.

Results

Theorem 1

If G is the $n \times n$ chess graph on the torus, $n \geq 5$, with all cops being knights and the robber being on foot, then $c(G) = 3$.

For this variation, assume all the cops moves as knights and the robber is on foot. To show that 3 cops can catch the robber, consider Figure 2. It's clear that if the cops move around the graph, keeping this rectangular shape, that they will easily be able to "trap" the robber in the rectangle. The cops are able to protect all but five of the vertices (vertices 1, 4, 7, and their symmetric opposites, in Figure 3) in this rectangular configuration. Considering each of these cases, we show that the cops will catch the robber.

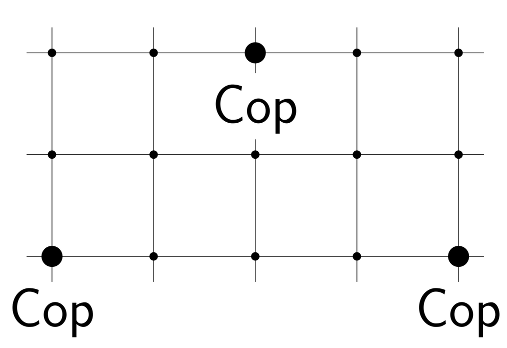

Figure 2: Positioning cops to "chase" the robber

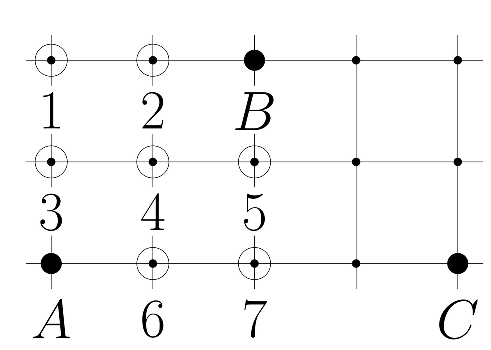

Figure 3: Cops A, B, C and robber locations (circled) to consider

Theorem 2

If all cops are on foot except one that is a chief and the robber is m-speedy, then

$$c(G) = 2\left\lceil \frac{m}{3} \right\rceil + 1.$$

Now, suppose all cops are on foot except a single cop who is a chief, with the robber being m-speedy. We can assume that the chief always protects the entire column in which the robber is located. In the row the robber is in, there are m vertices on either side of the robber that the robber could potentially move to. Since each cop on foot protects 3 vertices in a given row, and if there are f cops protecting the vertices on one side (the planar view of G) of the robber, we must have $3f \geq m$. Thus, the minimal number of cops on foot on either side of the robber must be $\left\lceil \frac{m}{3} \right\rceil$, placing a lower bound on the cop number of G of $c(G) \geq 2\left\lceil \frac{m}{3} \right\rceil + 1$. Figure 4 justifies why this is an upper bound on the cops.

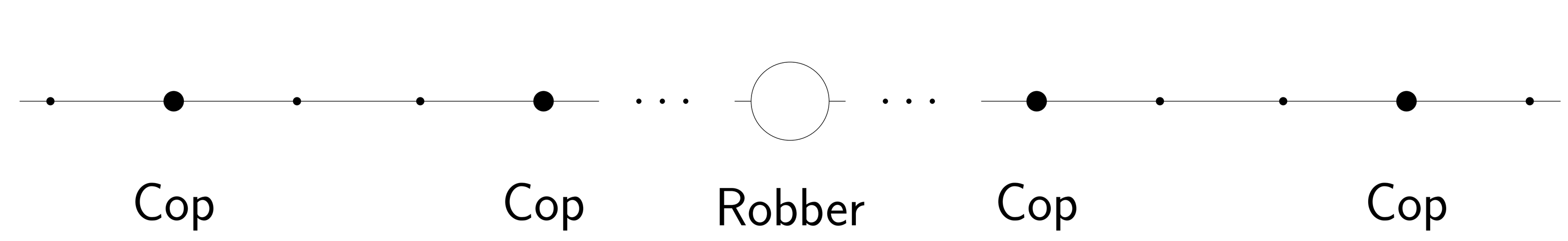

Figure 4: f cops on either side of the robber

Questions

On an $n \times n$ toroidal graph:

- What if the cops consist of a single chief and the rest as knights?
- What if some of the cops are speedy [1]?
- If the robber moves as a rook in chess, is a knight, or can move like a chief, how does the cop number change?
- What if we address each situation from a lazy cop perspective [2,4]?
- What if cops are only allowed to move a certain number of times before they must "take a mandatory break?"

References

[1] N. Alon, A. Mehrabian, Chasing a fast robber on planar graphs and random graphs, *J. Graph Theory* **78**(2) (2015) 81–96.

[2] D. Bal, A. Bonato, W. B. Kinnersley, P. Pralat, Lazy cops and robbers played on random graphs and graphs on surfaces, *J. Comb.* **7**(4) (2016) 727–642.

[3] A. Bonato, R. Nowakowski, *The Game of Cops and Robbers on Graphs*. American Mathematical Society, Providence, RI, 2011.

[4] B. Sullivan, N. Townsend, M. Werzanki, An introduction to lazy cops and robbers on graphs, *College Math. J.* **48**(5) (2017) 322-333.

More Information

You can find the full paper on the Arxiv at:

ID: 1810.10577

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