Computations of the (Weak) Global Dimension of a Ring

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Category Theory

- A **ring**, R, is an abelian group, $(R,+)$, together with another associative binary operation (which we represent here with "$\cdot$") satisfying

$$a\cdot(b+c)=a\cdot b+a\cdot c \qquad (a+b)\cdot c=a\cdot c+b\cdot c.$$

Example $\mathbb{Z}$ is a commutative ring.

- Let R be a ring. A left R**-module** is an additive abelian group, A, together with a function $R\times A\to A$, denoted by $r\cdot a\in A$, such that for any $r,s\in R$ and $a,b\in A$:
  (i) $r(a+b)=ra+rb$.
  (ii) $(r+s)a=ra+sa$.
  (iii) $r(sa)=(rs)a$.

Example $2\mathbb{Z}$ is a module over the ring $\mathbb{Z}$.

- A **chain complex** is a sequence of R-modules,

$$\cdots\to C_{p+1}\xrightarrow{d_{p+1}} C_p\xrightarrow{d_p} C_{p-1}\to\dots,$$

where each differential, d_i, is an R-module homomorphism and $d_i\circ d_{i+1}=0$. Often, we simply write $d^2:=d\circ d=0$.

- A **category**, $\mathcal{C}$, is a collection of objects and morphisms such that:
  Each morphism, $X\xrightarrow{f}Y$, has specified domain and codomain objects.
  1. Each object has an identity morphism $id_X: X\to X$.
  2. For any two morphisms $f:X\to Y$, $g:Y\to Z$ where the codomain of f equals the domain of g, there is a composite morphism, $X\xrightarrow{gf}Z$.

  Every category is subject to the following two axioms:
  1. For every $f:X\to Y$, $id_y f=f id_x=f$.
  2. For any composable triple of morphisms f, g, h, we have

$$h(gf)=(hg)f=hgf$$

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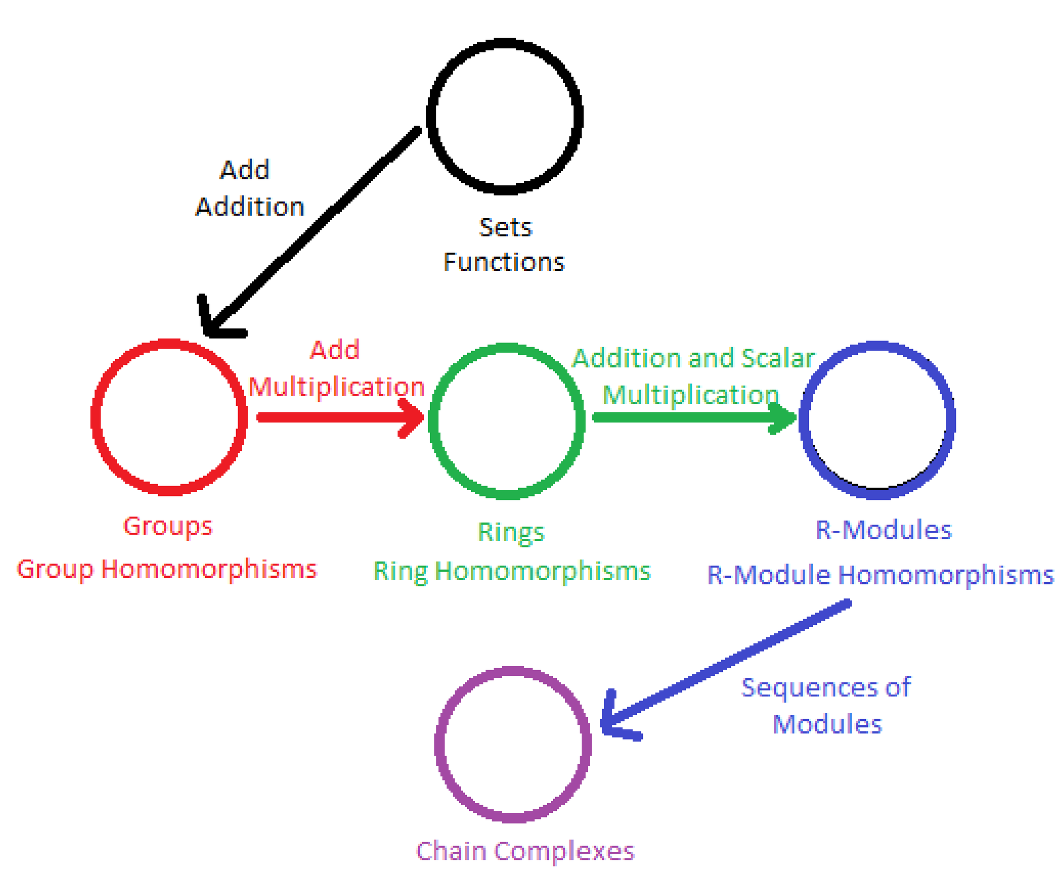

- A **functor** $F:\mathcal{C}\to\mathcal{D}$ is a morphism between categories that sends objects to objects and morphisms to morphisms:

$$\begin{array}{ccc} c & \longrightarrow & Fc=d \\ \downarrow f & & \downarrow Ff \\ c' & \longrightarrow & Fc'=d' \end{array}$$

satisfying
1. For any composable pair $f,g\in\mathcal{C}$, $Fg\circ Ff=F(g\circ f)$.
2. For each object $c\in\mathcal{C}$, $F(id_c)=id_{F_c}$.

Free and Projective Modules

- An R-module F is **free** on the subset A of M if every nonzero element m of M can be uniquely written as a linear combination:

$$m=r_1a_1+r_2a_2+\cdots+r_na_n,\ r_i\in R,\ a_i\in A.$$

We call A a basis for F.
Example (free $\mathbb{Z}$ module): $\mathbb{Z}$ and $\mathbb{Z}\times\mathbb{Z}$.
Example (free R module): $R\times R$.

Universal Property of Free Modules:

$$\begin{array}{ccc} A & \xrightarrow{\text{inclusion}} & F(A) \\ & \searrow^{\varphi} & \downarrow \Phi \\ & & M \end{array}$$

Prop: We have a functor, $Set\xrightarrow{Free}Mod_R$.

Prop: When A is a finite set, $a_1,a_2,\dots,a_n$, then $F(A)=Ra_1\times Ra_2\times\cdots\times Ra_n\cong R^n$.

- A **free resolution** of the R-module M is an exact sequence

$$\cdots\to F_2\xrightarrow{d_2}F_1\xrightarrow{d_1}F_0\xrightarrow{\epsilon}M\to 0$$

where each F_i is free.

- An R-module P is **projective** if whenever f is surjective and h is any map, there exists a map g such that the following diagram commutes.

$$\begin{array}{ccccc} & & P & & \\ & \swarrow^{g} & \downarrow h & & \\ A & \xrightarrow{f} & B & \longrightarrow & 0 \end{array}$$

Example Let $R=\mathbb{Z}/6\mathbb{Z}=\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}$. Then $M_1=\mathbb{Z}/2\mathbb{Z}$ and $M_2=\mathbb{Z}/3\mathbb{Z}$ are projective, but $R\ncong M_1$ and $R\ncong M_2$. So M_1 and M_2 are not free.

Projective Resolution

There are two equivalent definitions for projective resolution:

1. A **projective resolution** is a quasi-isomorphism in $Ch_{\geq 0}$:

$$\begin{array}{ccccccccc} \cdots & \to & P_3 & \to & P_2 & \to & P_1 & \to & P_0 \\ & & \downarrow\phi_3 & & \downarrow\phi_2 & & \downarrow\phi_1 & & \downarrow\varepsilon \\ \cdots & \to & 0 & \to & 0 & \to & 0 & \to & M \end{array}$$

where $M_i=M$ if $i=0$ and $M_i=0$ otherwise.

2. A **projective resolution**, $P^\bullet$, of an R-module, M, is an exact sequence

$$\cdots\xrightarrow{d_4}P_3\xrightarrow{d_3}P_2\xrightarrow{d_2}P_1\xrightarrow{d_1}P_0\xrightarrow{\varepsilon}M\to 0$$

where each P_i is a projective R-module.

Example: $\cdots\to 0\to\mathbb{R}\xrightarrow{d_1}\mathbb{R}^2\xrightarrow{\varepsilon}\mathbb{R}\to 0$
Or alternatively,

$$\begin{array}{ccccccccc} \cdots & \to & 0 & \to & 0 & \to & \mathbb{R} & \to & \mathbb{R}^2 \\ & & \downarrow\phi_3 & & \downarrow\phi_2 & & \downarrow\phi_1 & & \downarrow\varepsilon \\ \cdots & \to & 0 & \to & 0 & \to & 0 & \to & \mathbb{R} \end{array}$$

Abstract

Groups, Rings, and R-modules form the popular categories in which Modern Algebra is studied. One way that algebraists study Rings is by computing their (weak) global dimension which is an invariant of the Ring in question. This poster first walks through the familiar calculations of the projective dimension of a given R-module. Next, we focus on computing the (weak) global dimension of certain well-known rings before moving onto some more exotic cases. At the same time, we demonstrate how to use the language of category theory to describe these definitions when possible.

Dimension Theory

- The **projective dimension** of M over R, denoted $\text{pd}_R M$, is the smallest non-negative integer n for which there is a projective resolution

$$0\to P_n\to P_{n-1}\to\cdots\to P_1\to P_0\to M\to 0.$$

Essentially, $\text{pd}_R M=\inf_{P^\bullet}(\text{length}(P^\bullet))$.

- It follows that if M is projective, then $\text{pd}_R M=0$.
- The **global dimension** of R is defined by $\text{gl dim}\,R=\sup\{\text{pd}_R M : M \text{ is an } R\text{-module}\}$.
- A module is called **semisimple** if every submodule of M is a summand. A ring is semisimple if it is a semisimple module over itself.

Result: If a ring R is semisimple then $\text{gl dim}\,R=0$.

Example: $\mathbb{R}\times\mathbb{R}=\{(a,b):a,b\in\mathbb{R}\}$.

- A **hereditary** ring is one in which all submodules of projective modules are projective.

Result: A ring R is hereditary if and only if $\text{gl dim}\,R\leq 1$.

Example: $\mathbb{Z}[\sqrt{-5}]=\{a+b\sqrt{-5}:a,b\in\mathbb{Z}\}$

Future Study

- Understand rings with $\text{gl dim}\,R>1$.
- Instead of using projective modules we can use the larger class of flat modules to investigate the weak global dimension of rings.
- Investigate connection between the Ext functor and ring properties.

References

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