Fighting Gerrymandering with Math: Evaluating the Declination and a New Metric

Andrea Padilla, Dr. Ellen Veomett, Saint Mary's College of California

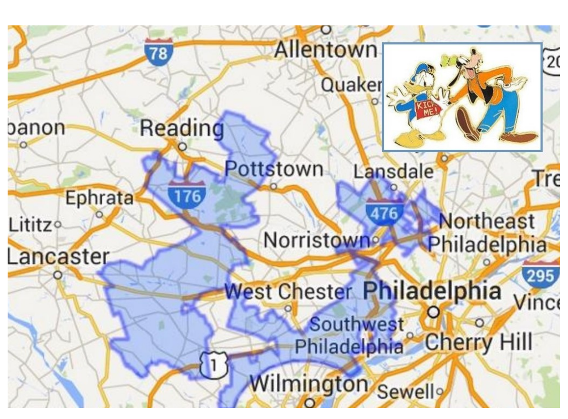
Introduction

Abstract

Gerrymandering is a largely debated issue in our current politics. There are many mathematical metrics that have been created in an attempt to detect gerrymandering. They all have certain advantages and disadvantages. The metric most discussed in this project is the declination, created by Greg Warrington. It has advantages, disadvantages, and implications that will be discussed. A new metric will be shortly proposed that attempts to compensate for the shortcomings of other metrics.

What is gerrymandering?

For the purposes of electing representatives, areas of land, states specifically, must be divided into districts. Gerrymandering occurs when the boundaries of a district are manipulated to favor a groups personal agenda. This is a current issue as there have been several suspicions of gerrymandering debated in the courts in recent years. A popular example is Pennsylvania's 7th congressional district created in 2011, or as it is pictured below, Goofy Kicking Donald, which has now been redistricted. As a result of these court cases and suspicions, many metrics have been created in an attempt to mathematically assess whether gerrymandering has occurred. These tools use shapes, voting data, and more to aid court cases in an attempt for just representation.



Packing and Cracking

Gerrymandering is done by packing and/or cracking ones opponents. Packing is done by putting all of the opponent's votes into a small amount of districts, packing their votes into these districts. The opponent then wins a small amount of districts by an overwhelming majority. Cracking is done by spreading the opponents votes over many districts, so that they lose most of them. In a lot of cracked districts, the opponent suffers a very narrow loss. The opponent has a surplus of losing votes. Both of these techniques succeed in essentially taking the opponents votes and distributing them so that they are wasted.

In the picture below from Kris Tapp's paper, "Measuring Political Gerrymandering" , districts are divided by lines, and the red and blue dots represent voters of different parties. Figure 1 shows proportional representation, but all districts are packed because each party wins by an overwhelming majority. In the second figure, the blue party is cracked because they suffer near losses in all districts and win none. Figure 3 shows the red party is packed in the 2 left districts, and cracked in the right upper and lower districts, while the blue party is packed in the right middle district.

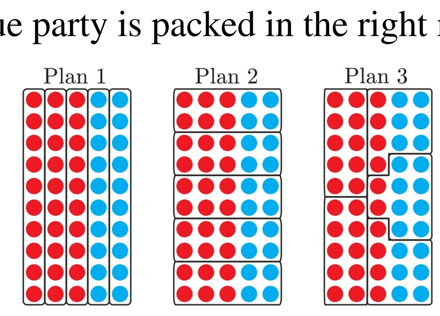


FIGURE 1. Three ways to divide 50 voters into 5 districts

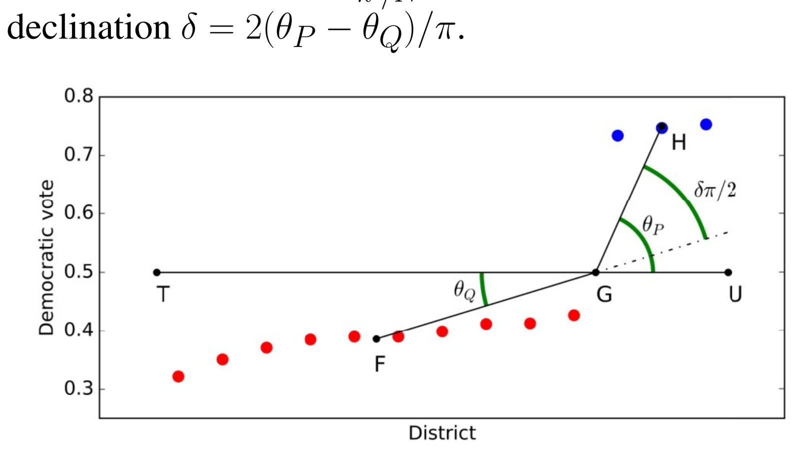
Declination

Definition

Warrington first proposed the declination in his paper, "Introduction to the Declination Function for Gerrymandering." In short, his metric is concerned with vote distribution among districts and "treats asymmetry... as indicative of gerrymandering" (1).

The declination is a highly visual metric, and is broken down by Warrington in "Quantifying Gerrymandering Using the Vote Distribution" (4). Essentially, a line is drawn at $y = 0.5$. Each district's vote share is plotted as a dot, in increasing order of vote share for party P, with districts that party P has lost plotted below $y = 0.5$, as the vote share in that district is less than 50%. Order the N districts in order of vote share, then k is the number of districts lost by party P and $k' = N - k$ is the number of districts won by party P. The districts that party P has won are plotted as dots above $y = 0.5$ with their respective vote shares as y-values.The x-coordinate of each district $p_i = \frac{i}{N} - \frac{1}{2N}$.

A point G is plotted at the center $(\frac{k}{N}, 1/2)$, and let $T = (0, \frac{1}{2})$, $U = (1, \frac{1}{2})$. Let $\overline{y}$ be the average losing vote share and $\overline{z}$ be the average winning vote share. Then $F = (\frac{k}{2N}, \overline{y})$ is the center of mass of the districts which party P is losing and $H = (\frac{k}{N} + \frac{k'}{2N}, \overline{z})$ is the center of mass of the districts won by party P. Finally, let $\theta_P = \arctan(\frac{2\overline{z}-1}{k'/N}) = \angle HGU$ and $\theta_Q = \arctan(\frac{1-2\overline{y}}{k/N})$. Then the declination $\delta = 2(\theta_P - \theta_Q)/\pi$.



Does It Always Work?

Greg Warrington claims that the declination increases in response to gerrymandering, and in most cases it does. However, we have found cases where the declination does not detect packing or cracking. Furthermore, these examples have districts with ratios of highest to lowest voter turnout that are comparable to real ratios, meaning that these examples could very well happen and go undetected by the declination.

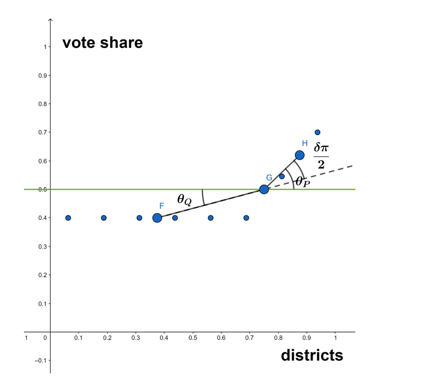
The first table below shows the initial election and declination. The second table shows that the votes from the last district have been shifted so that Party A is now losing that district and is cracked in the other districts. The declination does not increase as Greg claims it will, but decreases.

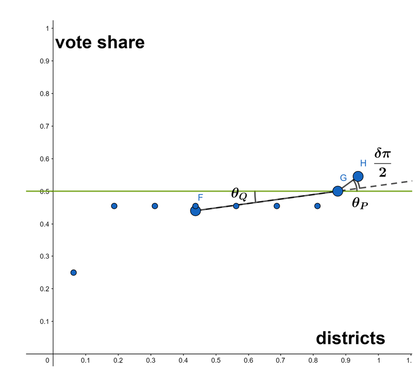
Party A	4	4	4	4	4	4	6	7
Party B	6	6	6	6	6	6	5	3

$\delta = 0.328255$

Party A	5	5	5	5	5	5	6	1
Party B	6	6	6	6	6	6	5	3

$\delta = 0.292679$





The fact that the declination doesn't always work the way that it is meant to doesn't make it obsolete. We found in the declination that there are certain implications for "ideal" elections.

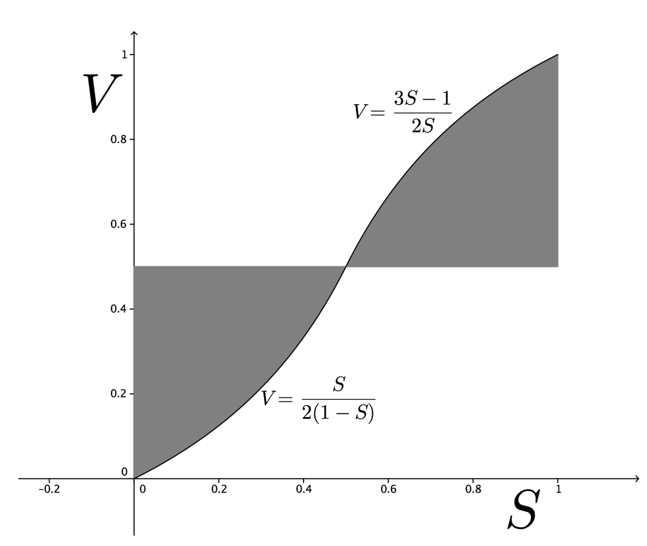
Elections with $\delta = 0$

Warrington doesn't say this explicitly, but it is implied that in a "fair" election, the declination is equal to zero, there is no asymmetry in vote share. For our purposes we will assume that voter turnout in all districts is equal. Let the seat share of party P $S = \frac{k'}{N} = \frac{N-k}{N}$, then by setting the declination equal to zero we have the relationship $S = \frac{1-2\overline{z}}{\overline{y}-\overline{z}}$. Now to make this a relationship in only two variables, let the vote share of party P be $V = (1-S)\overline{y} - S\overline{z}$, since S represents the percentage of districts won by party P.

Theorem. *Given a declination of 0 and seat share S, then the vote share V can take on any value, based on what S is:*

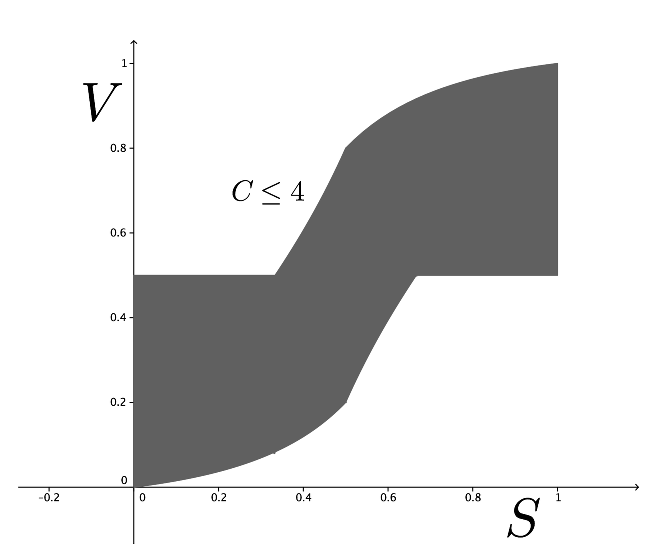
$$\text{if } 0 < S \leq 0.5 \implies \frac{S}{2(1-S)} \leq V \leq 0.5$$
$$\text{if } 0.5 < S < 1 \implies 0.5 \leq V \leq \frac{3S-1}{2S}$$

This theorem is illustrated below. Given any seat share S, a vote share V can be chosen in the given range such that an election with a declination of 0 occurs.



Possible seat and vote share options for equal turnout

Now suppose turnout is not equal in all districts. Let C be the ratio of highest turnout to lowest turnout. An analogous theorem to the one above can be made, but it has far more restrictions. The visual, however, translates similarly to the original and is pictured below for $C = 4$.



Seat and vote share options for turnout ratio $C = 4$

New Metric

The aim of the new metric is to take into account the geometry of a state's districts and the election data to determine if specific districts could have been packed and/or cracked. One advantage of this metric is that it is the only metric to account for election data and geographic information. It also allows us to see exactly which districts have been gerrymandered, which is an issue the courts have had with other metrics used.

Definition

Given a districting plan with N districts, a graph is made where each district is a vertex and an edge, a line connecting two vertices, is drawn if the districts share a boundary. Each district D_i with percentage vote share p_i for party P will have a packing count P_i, outward cracking count C_i^o, and inward cracking count C_i^i. Looking at each D_i for $1 \leq i \leq N$:

If $0 \leq p_i < 0.4$, look at the districts which D_i shares an edge with. For each district D_j with $0.4 \leq p_j < 0.5$, add one point to C_i^o. We are counting the districts which party P could have won if we swapped 10% of the party P votes in D_i with 10% of the party Q votes in D_j.

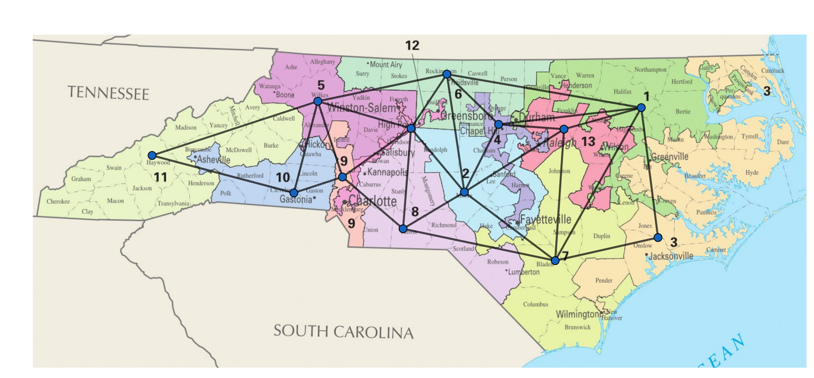
If $0.4 \leq p_i < 0.5$, for each surrounding district D_j with p_j, if $0 \leq p_j \leq 0.4$ or $p_j > 0.6$, add one point to C_i^i. We are counting the surrounding districts which could trade 10% of their party P votes with 10% of D_i's party Q votes without changing the election outcome.

For $p_i \geq 0.6$, if D_j has $0.4 \leq p_j < 0.5$, add one point to P_i.

Examples

The first table shown below is the 2016 presidential election data for North Carolina by district. Below that is a graph of North Carolina's districts, including the edges shared by neighboring districts. The final table is shows each districts counts of packing and cracking by the new metric.

District	Democratic	Republican
District 1	68.88%	31.12%
District 2	45.04%	54.96%
District 3	37.89%	62.11%
District 4	70.75%	29.25%
District 5	40.95%	59.05%
District 6	42.46%	57.54%
District 7	40.92%	59.08%
District 8	42.28%	57.72%
District 9	44.03%	55.97%
District 10	37.37%	62.63%
District 11	34.98%	65.02%
District 12	70.66%	29.34%
District 13	45.17%	54.83%



Graph of North Carolina's Congressional districts (2014)

District k	P_k	C_k^o	C_k^i
District 1	3	0	0
District 2	0	4	6
District 3	0	1	0
District 4	3	0	0
District 5	0	2	5
District 6	0	2	5
District 7	0	3	5
District 8	0	3	4
District 9	0	2	4
District 10	0	2	0
District 11	0	1	0
District 12	5	0	0
District 13	0	2	4

The alternative metric suggests there has been gerrymandering against the Democrats in all districts. To highlight some of the larger counts, district 2 has a total cracking count of 10. District 7 has a cracking count of 8 and districts 5,6, and 8 have cracking counts of 7. It should also be noted that by the rules of this metric, the Republican party has not been packed or cracked in any district; this metric suggests there has been no gerrymandering against the Republican party in this election.

References

Kris Tapp. "Measuring Political Gerrymandering," 7, 2018. Greg Warrington. "Quantifying Gerrymandering Using the Vote Distribution," 4, 2017. Greg Warrington, "Introduction to the Declination Function for Gerrymandering," 1, 2018.