Exploring Derivatives of Square Knots

Rilee Potter

Department of Mathematics at the University of Nebraska-Lincoln
Advised by Dr. Alexander Zupan

Research Question

Which knots occur as components of a Casson-Gordon derivative of the square knots?

Introduction

The exploration of derivatives of knots involves studying curves on Seifert surfaces of square knots.

Seifert Surface

A Seifert surface is defined as a surface bounded by the knot or link as shown in Figure 1.

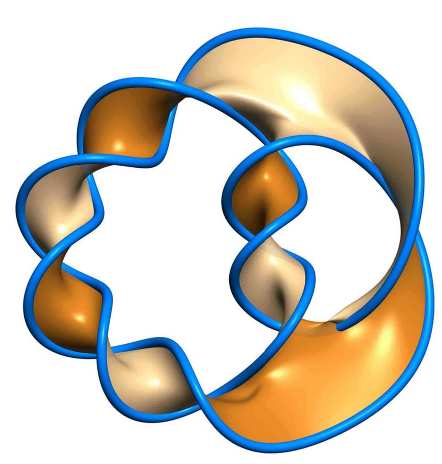

Figure 1: Seifert Surface [1]

Specifically, we are looking at curves on the Seifert surface of the square knot, pictured in Figure 2.

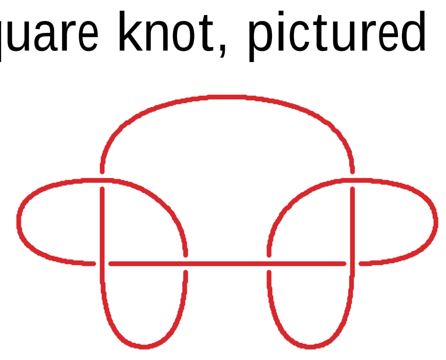

Figure 2: Square Knot

The derivatives are specifically known as Casson-Gordon derivatives.

Casson-Gordon Derivative

This can be defined as a link contained in the genus two Seifert surface for the square knot obtained from a function that associates a link to each ratio with an odd denominator.

We let $L(\frac{a}{b})$ denote the 3-component link associated to the ratio $\frac{a}{b}$.

Main Objectives

- Collect data about each knot on the Casson-Gordon derivative
- Analyze the data collected for possible patterns
- Predict what comes from this data

Analysis Methods

The process of collecting data involves the following steps:

1. Picking a ratio and transferring this ratio onto the surface as shown in Figure 3.

Figure 3: Drawings of $L(\frac{0}{1})$ and $L(\frac{2}{7})$

2. Counting intersection numbers of the components with the sides of the hexagon
3. Transferring components onto the Seifert surface for the square knot

Figure 4: Drawings of $L(\frac{0}{1})$ and $L(\frac{2}{7})$ Casson-Gordon Derivatives

As you can see in Figure 4, the knots can get quite complicated. The computer program SnapPy can be implemented to obtain certain data through these complications.

4. Determining the knots created through SnapPy
5. Simplifying these knots and obtaining their crossing number and volume

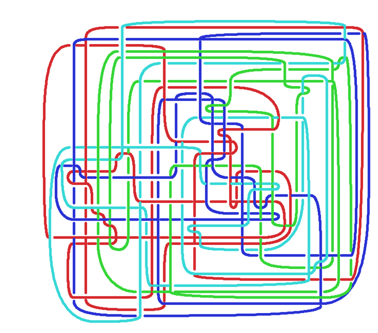

Figure 5: Simplification of $L(\frac{2}{7})$

Results

Results come from the collection of crossing numbers and the values of the volume of each of the knot components that I have drawn and then implemented into SnapPy. Conjecture 1 provides a possible explanation for the patterns in Table 1.

Conjecture 1

$L(\frac{a}{an-1})$ and $L(\frac{a}{an+1})$ have a component in common.

a	2	3	4	5	6	7	8
	$\frac{2}{5},\frac{2}{7}$	$\frac{3}{5},\frac{3}{7}$	$\frac{4}{7},\frac{4}{9}$	$\frac{5}{9},\frac{5}{11}$	$\frac{6}{11},\frac{6}{13}$	$\frac{7}{13},\frac{7}{15}$	$\frac{8}{15},\frac{8}{17}$
	$\frac{2}{7},\frac{2}{9}$	$\frac{3}{11},\frac{3}{13}$	$\frac{4}{11},\frac{4}{13}$	$\frac{5}{19},\frac{5}{21}$	$\frac{6}{17},\frac{6}{19}$	$\frac{7}{27},\frac{7}{29}$	$\frac{8}{23},\frac{8}{25}$
	$\frac{2}{9},\frac{2}{11}$	$\frac{3}{17},\frac{3}{19}$	$\frac{4}{15},\frac{4}{17}$	$\frac{5}{29},\frac{5}{31}$	$\frac{6}{23},\frac{6}{25}$	$\frac{7}{41},\frac{7}{43}$	$\frac{8}{31},\frac{8}{33}$
	$\frac{2}{11},\frac{2}{13}$	$\frac{3}{17},\frac{3}{19}$	$\frac{4}{19},\frac{4}{21}$	$\frac{5}{39},\frac{5}{41}$	$\frac{6}{29},\frac{6}{31}$	$\frac{7}{55},\frac{7}{57}$	$\frac{8}{39},\frac{8}{41}$

Table 1: Data for Conjecture 1

Each pair of knots in Table 1 has a component that is isotopic. The pairs of knots in black we know follow the pattern stated in Conjecture 1 and the pairs of knots in red we predict to follow the pattern. For example, see Figure 6, below.

b	5	7	9	11	13	15
	$\frac{2}{5},\frac{3}{5}$	$\frac{2}{7},\frac{5}{7}$	$\frac{4}{9},\frac{5}{9}$	$\frac{5}{11},\frac{6}{11}$	$\frac{5}{13},\frac{8}{13}$	$\frac{7}{15},\frac{8}{15}$
	$\frac{1}{5},\frac{4}{5}$	$\frac{3}{7},\frac{4}{7}$	$\frac{1}{9},\frac{8}{9}$	$\frac{4}{11},\frac{7}{11}$	$\frac{4}{13},\frac{9}{13}$	
		$\frac{1}{7},\frac{6}{7}$	$\frac{3}{9},\frac{6}{9}$	$\frac{3}{11},\frac{8}{11}$	$\frac{6}{13},\frac{7}{13}$	

Table 2: Data for Conjecture 2

Figure 6: Simplifications of the Isotopic Component of $L(\frac{2}{5})$ and $L(\frac{2}{7})$

Figure 5 shows images of the isotopic components of the $L(\frac{2}{5})$ and $L(\frac{2}{7})$ knots which corresponds to the first pair listed in Table 1.

Conjecture 2

$L(\frac{a}{b})$ and $L(\frac{b-a}{b})$ are equivalent.

Table 2 corresponds with the pattern described in Conjecture 2. This relevant pattern allows for each knot pair in Table 2 to have all isotopic components and the ratios of the links $L(\frac{a}{b})$ have a sum of one. For example, take a look again at the single component of $L(\frac{2}{7})$ on the right hand side in Figure 6. $L(\frac{2}{7})$ is isotopic, as a link, to $L(\frac{5}{7})$.

Forthcoming Research

Extending this research involves analyzing the data I am collecting for more patterns and continuing to look at more derivatives. SnapPy is the Python based computer program I am using to draw these complicated knots and collect data. I am currently trying to write another program to involve in analyzing more Casson-Gordon derivatives on a different Seifert surface.

Acknowledgements

Prof. Nathan Dunfield from UIUC wrote the program that I use in SnapPy.

References

[1] Clay Cordova, Sam Espahbodi, Babak Haghighat, Ashwin Rastogi, and Cumrun Vafa.
Tangles, Generalized Reidemeister Moves, and Three-Dimensional Mirror Symmetry.
Journal of High Energy Physics, 2012.

University of Nebraska Lincoln