Exploring Derivatives of Square Knots

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Introduction
The exploration of derivatives of knots involves studying curves on Seifert surfaces of square knots.

Seifert Surface
A Seifert surface is defined as a surface bounded by the knot or link as shown in Figure 1.

Figure 1: Seifert Surface [1]

Specifically, we are looking at curves on the Seifert surface of the square knot, pictured in Figure 2.

Figure 2: Square Knot

The derivatives are specifically known as Casson-Gordon derivatives.

Casson-Gordon Derivative
This can be defined as a link contained in the genus two Seifert surface for the square knot obtained from a function that associates a link to each ratio with an odd denominator.

We let $L(\frac{a}{b})$ denote the 3-component link associated to the ratio $\frac{a}{b}$.

Main Objectives
- Collect data about each knot on the Casson-Gordon derivative
- Analyze the data collected for possible patterns
- Predict what comes from this data

Analysis Methods
The process of collecting data involves the following steps:
- Picking a ratio and transferring this ratio onto the surface as shown in Figure 3.
- Counting intersection numbers of the components with the sides of the hexagon
- Transferring components onto the Seifert surface for the square knot

Figure 3: Drawings of $L(\frac{a}{b})$ and $L(\frac{c}{d})$

As you can see in Figure 4, the knots can get quite complicated. The computer program SnapPy can be implemented to obtain certain data through these complications.
- Determining the knots created through SnapPy
- Simplifying these knots and obtaining their crossing number and volume

Figure 4: Drawings of $L(\frac{a}{b})$ and $L(\frac{c}{d})$ Casson-Gordon Derivatives

Results
Results come from the collection of crossing numbers and the values of the volume of each of the knot components that I have drawn and then implemented into SnapPy. Conjecture 1 provides a possible explanation for the patterns in Table 1.

Conjecture 1

Each pair of knots in Table 1 has a component that is isotopic. The pairs of knots in black we know follow the pattern stated in Conjecture 1 and the pairs of knots in red we predict to follow the pattern. For example, see Figure 6, below.

Figure 5: Simplification of $L(\frac{a}{b})$

Table 1: Data for Conjecture 1

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Table 2: Data for Conjecture 2

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Conjecture 2

$L(\frac{a}{b})$ and $L(\frac{c}{d})$ are equivalent.

Forthcoming Research
Extending this research involves analyzing the data I am collecting for more patterns and continuing to look at more derivatives. SnapPy is the Python based computer program I am using to draw these complicated knots and collect data. I am currently trying to write another program to involve in analyzing more Casson-Gordon derivatives on different Seifert surfaces.

Acknowledgements
Prof. Nathan Dunfield from UIUC wrote the program that I use in SnapPy.

References
Tangles, Generalized Reidemeister Moves, and Three-Dimensional Mirror Symmetry.