Classifying Polynomial and Rational Solution of a Functional Equation

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Abstract
The research aims to investigate solutions of the functional equation

\[ U(z)f(z) + D(z)f(U(z)) = 0 \]

For particular linear fractional maps \( U(z) \), we checked different types of functions for \( f \) (polynomials and rational functions) to see whether they solve the functional equation or not. In some scenarios, we have classified the solutions or have shown a class of functions that are not solutions.

Introduction
Definition. Given an analytic function \( \varphi \), the composition operator \( C_\varphi \) is defined by \( C_\varphi f = f \circ \varphi \), where \( \varphi \) is referred to as the symbol of the composition operator.

\( \varphi \) is defined as a rational function of degree two with real coefficients, \( \varphi \) has the form

\[ \varphi(z) = \frac{a_1z^2 + b_1z + c_1}{a_2z^2 + b_2z + c_2} \]

where \( c_2 \neq 0 \) and there’s at least one nonzero quadratic term. Furthermore, \( \varphi \) is in the Hardy space.

The Hardy space, \( H^2 \), is the Hilbert space of analytic functions for which \( \varphi(z) \) is always represented by one of following \( m \) forms

\[ \sum_{j=0}^{m} \psi_j(z) = 0 \]

Abstract

Method

Theorem 1. Let \( \varphi \) be a non-constant rational map, and let \( C_\varphi \) act on \( H^2 \). Set

\[ \sigma(z) = \frac{1}{1 - \varphi(z)} \]

and \( \psi(z) = \sigma(\varphi(z)) \). Then,

\[ C_\varphi f(z) = \frac{1}{1 - \varphi(z)} \sum_{j=1}^{\infty} \psi_j(z) f(\sigma(z)) \]

Theorem 2. There exists a function \( U(z) \), such that \( U(z) f(z) = \sigma(\varphi(z)) \), and \( U(z) \psi(z) = \varphi(z) \), where \( \sigma(z) \) and \( \psi(z) \) are branches of \( \varphi(z) \) and \( \varphi(z) \), respectively.

\[ U(z) = \frac{(\varphi_2 - \varphi_1)z + (\varphi_2 \varphi_1 - \varphi_1 \varphi_2)z}{(\varphi_2 - \varphi_1)z + (\varphi_2 \varphi_1 - \varphi_1 \varphi_2)z} = \frac{a_1z^2 + b_1z + c_1}{a_2z^2 + b_2z + c_2} \]

The kernel of \( C_\varphi f \) is a subset of \( H^2 \) such that \( f \in ker(C_\varphi f) \), then \( C_\varphi f(z) = 0 \).

Theorem 3. Let \( \varphi \) be a rational map of degree two mapping \( D \) onto \( D \) and let \( C_\varphi \) act on the Hardy space. For \( U \) as in Theorem 2, \( f \in ker(C_\varphi f) \) if and only if

\[ U(z)f(z) + D(z)f(U(z)) = 0 \]

Solve the system equation in terms of \( n \), \( \alpha \) are free variables.

Then, the rest of polynomial becomes to the odd degree polynomial which is already solved. Therefor, when \( f(z) \) is a third degree polynomial, the coefficient of highest power of \( z \) is free.

The coefficients of each power of \( z \) in equation (5) produce the following system of equations. Generally, these equations consist of a general system equation shown as follow:

\[ \begin{align*}
0 &= -\alpha_1 |(-1)^3 + 1|^m z^{m+1} \\
0 &= \alpha_0 m |(-1)^m - 1|^m z^m + \alpha_{m-1} |(-1)^m - 1|^m (m - 1) z^{m-1} + \alpha_{m-2} |(-1)^m - 1|^m (m - 2) z^{m-2} + \cdots + \alpha_1 |(-1)^m - 1|^m \frac{1}{1 + z} z^1 + \alpha_0 |(-1)^m - 1|^m \frac{1}{1 + z} z^0 \\
0 &= -\alpha_0 m^m z + \alpha_{m-1} m^{m-1} z + \cdots + \alpha_2 m (m - 2) z^2 + \alpha_1 (m - 1) z + \alpha_0 m \\
0 &= \alpha_0 m^m z + \alpha_{m-1} m^{m-1} z + \cdots + \alpha_2 m (m - 2) z^2 + \alpha_1 (m - 1) z + \alpha_0 m \\
\end{align*} \]

From equation (6), the highest power of \( z \) is \( n + 1 \), the corresponding equation is

\[ \begin{align*}
0 &= -\alpha_1 |(-1)^n + 1|^m z^{m+1} \\
0 &= \alpha_0 m |(-1)^m - 1|^m z^m + \alpha_{m-1} |(-1)^m - 1|^m (m - 1) z^{m-1} + \alpha_{m-2} |(-1)^m - 1|^m (m - 2) z^{m-2} + \cdots + \alpha_1 |(-1)^m - 1|^m \frac{1}{1 + z} z^1 + \alpha_0 |(-1)^m - 1|^m \frac{1}{1 + z} z^0 \\
0 &= -\alpha_0 m^m z + \alpha_{m-1} m^{m-1} z + \cdots + \alpha_2 m (m - 2) z^2 + \alpha_1 (m - 1) z + \alpha_0 m \\
0 &= \alpha_0 m^m z + \alpha_{m-1} m^{m-1} z + \cdots + \alpha_2 m (m - 2) z^2 + \alpha_1 (m - 1) z + \alpha_0 m \\
\end{align*} \]

When \( n \) is odd, \( \alpha_0 \) is always a free variable, which indicate that there always exists a class of solutions for the coefficients of odd degree polynomial.

As shown in equation (7), when \( n \) is even, the coefficient of highest power of \( z \) is always \( 0 \). Therefore, when \( f(z) \) is an even degree polynomial, can’t be the solutions of equation (3). In some scenarios, we have classified the solutions or have shown a class of functions that are not solutions.

Acknowledgments

This research was conducted during summer 2018 at Coe College under the direction of Professor Brittany Miller. This project was supported by Briale and Irene H. Perrine Faculty Fellowship at Coe College

Reference


Example

Let \( f(z) \) be a fourth degree polynomial \( f(z) = a_4 z^4 + a_3 z^3 + a_2 z^2 + a_1 z + a_0 \). Substitute \( f(z) \) into equation (4). Follow the method stated in the above section, the coefficients of each power of \( z \) produce the system equation shown as follow:

\[ \begin{align*}
0 &= a_4 \\
0 &= 4a_3 z + 3a_2 z^2 + 2a_1 z + a_0 \\
0 &= 12a_2 z + 6a_1 z + 2a_0 \\
0 &= 8a_1 z + 4a_0 \\
0 &= 4a_0 \\
\end{align*} \]

Figure 1: Possible \( f(z) \) when \( m = -0.5 \)
Figure 2: Possible \( f(z) \) when \( m = 2 \)

Solve the system equation in terms of \( a_1, a_2, a_3, a_4, a_5 \), the solution is \( a_1 = 0, a_2 = 1, a_3 = 0, a_4 = 0, a_5 = 0 \), where \( r_1 \) and \( r_2 \) are free variables.

This matches with the theorem that that coefficient of forth degree is \( 0 \). If we consider the solution as a third degree polynomial, the coefficient of highest power of \( z \) is a free variable.