Automorphisms and Characters of Finite Groups

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**Big Idea**

**Fixed Notation**

- $G = Sp_4(q)$ where $q$ is a power of an odd prime, $p$
- $H = \{ \text{diag}(a, a^{-1}, b, b^{-1}) \mid a, b \in \mathbb{F}_q^* \}$ a subgroup of $G$
- $\varphi^m_p$ is a "field automorphism" of $G$
- $\sigma$ is an automorphism of $\mathbb{Q}(e^{2\pi i/|G|})$

**Theorem**

Assume every $\varphi^m_p$-invariant member of $\text{Irr}(H)$ is also fixed by $\sigma$. Then every $\varphi^m_p$-invariant member of $\text{Irr}_{q-1}'(G)$ is also fixed by $\sigma$. 
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$Sp_4(q) = \{g \text{ is an invertible } 4 \times 4 \text{ matrix over } \mathbb{F}_q \mid g^T J g = J\}$

where $J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$
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- **Associativity**
  \[\forall a, b, c \in G, (a \star b) \star c = a \star (b \star c)\]
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  \[ \exists e \in G \text{ s.t. } \forall a \in G, a \star e = e \star a = a. \]
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Example: \(\mathbb{Z}_{12}\) under addition
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under the binary operation $\star$

Example: $\mathbb{Z}_{12}$ under addition
Non-Example: $\mathbb{Z}_{12}$ under multiplication
### Fixed Notation

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- \( \sigma \) is an automorphism of \( \mathbb{Q}(e^{2\pi i/|G|}) \)

So \( \text{Sp}_4(q) \) is a group?
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$Sp_4(q) = \{ g \text{ is an invertible } 4 \times 4 \text{ matrix over } \mathbb{F}_q \mid g^T J g = J \}$. 
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  Matrix multiplication is associative
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  $e = I$, the identity matrix since $I^TJI = J$
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- **Associativity**
  Matrix multiplication is associative
- **An identity element, $e$**
  
  $e = I$, the identity matrix since $I^T J I = J$
- **An inverse for every group element**
  Since $g^{-1}$ also satisfies the group definition: $(g^{-1})^T J (g^{-1}) = J$
  then every element has an inverse.
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- $G = \text{Sp}_4(q)$ where $q$ is a power of an odd prime, $p$
- $H = \{\text{diag}(a, a^{-1}, b, b^{-1}) \mid a, b \in \mathbb{F}_q^*\}$ a **subgroup** of $G$
- $\varphi_p^m$ is a “field automorphism” of $G$
- $\sigma$ is an automorphism of $\mathbb{Q}(e^{2\pi i/|G|})$

A subgroup $H$ is a subset of group elements of a group $G$ that is itself a group under the group operation.
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Example: The evens mod 12 forms a subgroup of $\mathbb{Z}_{12}$ under addition.
Non-Example: The odds mod 12 do not form a subgroup of $\mathbb{Z}_{12}$ under addition.
A diagonal matrix has zeros everywhere except the main diagonal.
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So diag($a, a^{-1}, b, b^{-1}$) is the diagonal matrix

$$\begin{bmatrix}
 a & 0 & 0 & 0 \\
 0 & a^{-1} & 0 & 0 \\
 0 & 0 & b & 0 \\
 0 & 0 & 0 & b^{-1}
\end{bmatrix}$$

With entries $a, b \in \mathbb{F}_q^*$
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But why is $H$ a subgroup of $G$?

If we let $g = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & a^{-1} & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & b^{-1} \end{bmatrix}$ and $J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$ then

$$g^T J g = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a^{-1} & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & b^{-1} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a^{-1} & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & b^{-1} \end{pmatrix} = J$$
But why is $H$ a subgroup of $G$?

Let $A = \begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_1^{-1} & 0 & 0 \\ 0 & 0 & b_1 & 0 \\ 0 & 0 & 0 & b_1^{-1} \end{bmatrix}$ and $B = \begin{bmatrix} a_2 & 0 & 0 & 0 \\ 0 & a_2^{-1} & 0 & 0 \\ 0 & 0 & b_2 & 0 \\ 0 & 0 & 0 & b_2^{-1} \end{bmatrix}$, then

$$AB = \begin{bmatrix} a_1a_2 & 0 & 0 & 0 \\ 0 & (a_1a_2)^{-1} & 0 & 0 \\ 0 & 0 & b_1b_2 & 0 \\ 0 & 0 & 0 & (b_1b_2)^{-1} \end{bmatrix}$$

Thus $H$ is closed under the group operation from $G$, so $H$ is a subgroup of $G$. 
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A **homomorphism** is a function of one group to another that preserves the group operation.

So for groups $(G, *)$ and $(\bar{G}, *)$, then for any $g_1, g_2 \in G$

$$f(g_1 * g_2) = f(g_1) * f(g_2)$$
A homomorphism is a function of one group to another that preserves the group operation. So for groups $(G, \star)$ and $(\bar{G}, \star)$, then for any $g_1, g_2 \in G$

$$f(g_1 \star g_2) = f(g_1) \star f(g_2)$$

An automorphism is a bijective homomorphism from a group $G$ onto itself.
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$\varphi^m_p$ is an automorphism of $Sp_4(q)$ that raises all entries of its operand to the power $p^m$.

Example: let $B_3(i, s) \in G$ such that $B_3(i, s) = \begin{bmatrix} \gamma^i & 0 & 0 & 0 \\ 0 & \gamma^{-i} & 0 & 0 \\ 0 & 0 & \gamma^s & 0 \\ 0 & 0 & 0 & \gamma^{-s} \end{bmatrix}$

(where $\gamma$ is a $q - 1$ root of 1 in $\mathbb{F}_q^*$)

(as defined in Srinivasan [3, Srinivasan 1968].)
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Observe $\varphi_p(B_3(i, s))$:

$\varphi_p \left( \begin{bmatrix} \gamma^i & 0 & 0 & 0 \\ 0 & \gamma^{-i} & 0 & 0 \\ 0 & 0 & \gamma^s & 0 \\ 0 & 0 & 0 & \gamma^{-s} \end{bmatrix} \right) = \begin{bmatrix} \gamma^{ip} & 0 & 0 & 0 \\ 0 & \gamma^{-ip} & 0 & 0 \\ 0 & 0 & \gamma^{sp} & 0 \\ 0 & 0 & 0 & \gamma^{-sp} \end{bmatrix} = B_3(ip, sp)$
**Representation**

A *representation* is a homomorphism $\rho$ from a group $G$ into a group of $n \times n$ invertible matrices with entries in $\mathbb{C}$.

$$\rho : G \rightarrow GL_n(\mathbb{C}) \text{ such that } \rho(gh) = \rho(g)\rho(h) \text{ for all } g, h \in G$$

(where $GL_n(\mathbb{C})$ is the group of $n \times n$ invertible matrices with entries in $\mathbb{C}$)
The *trace* of a matrix is the sum of its diagonal entries.

So \( \text{Tr}(A_{n \times n}) = a_{11} + a_{22} + \ldots + a_{nn} \).
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So \( \text{Tr}(A_{n\times n}) = a_{11} + a_{22} + \ldots + a_{nn} \).

So if 
\[
\begin{bmatrix}
a & 0 & 0 & 0 \\
0 & a^{-1} & 0 & 0 \\
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\end{bmatrix}
\]

Then \( \text{Tr}(h) = a + a^{-1} + b + b^{-1} \).
A character $\chi$ is the composition of the trace function with the representation of a group element.

$$\chi = \text{Tr} \circ \rho$$

$$\chi(g) = \text{Tr}(\rho(g))$$
Theorem

Assume every $\varphi^m_p$-invariant member of \text{Irr}(H) is also fixed by $\sigma$. Then every $\varphi^m_p$-invariant member of \text{Irr}_{(q-1)'}(G)$ is also fixed by $\sigma$. 

Recall...
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$\chi$ is irreducible if $\chi \neq \chi_1 + \chi_2$ for characters $\chi_1, \chi_2$

$\text{Irr}(H)$ is the set of irreducible characters of $H$. 
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$\text{Irr}(H)$ is the set of irreducible characters of $H$.

$\text{Irr}_{(q-1)'}(G)$ is the set of irreducible characters of $G$ such that $n$ is relatively prime to the quantity $(q - 1)$. 
Theorem
Assume every $\varphi^m_p$-invariant member of $\text{Irr}(H)$ is also fixed by $\sigma$.
Then every $\varphi^m_p$-invariant member of $\text{Irr}(q-1)'(G)$ is also fixed by $\sigma$.

Given the automorphism of $\varphi^m_p$ of $G$ and $\chi \in \text{Irr}(G)$, we can obtain a new irreducible character $\varphi^m_p \chi$ via

$$\varphi^m_p \chi(g) = \chi(\varphi^m_p(g))$$
Theorem
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Then every \( \varphi_p^m \)-invariant member of \( \text{Irr}(q^{-1})'(G) \) is also fixed by \( \sigma \).

\( \varphi_p^m \) is an automorphism of \( Sp_4(q) \) that raises all entries of its operand to the power \( p^m \). So \( \varphi_p^m(\chi(g)) = \chi(\varphi_p^m(g)) \).

Looking at \( \chi_8(k) \), we claim that \( \varphi_p(\chi_8(k)) = \chi_8(kp) \).
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Looking at $\chi_8(k)$, we claim that $\varphi_p(\chi_8(k)) = \chi_8(kp)$.

For example, we know that $\varphi_p(B_3(i, s)) = B_3(ip, sp)$.
Now let $\chi = \chi_8(k)$ and $g = B_3(i, s)$, where $\chi_8$ is a character of $G$. 

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Now let $\chi = \chi_8(k)$ and $g = B_3(i, s)$, where $\chi_8$ is a character of $G$.

Notice that $\chi(g) = (\tilde{\gamma}^{ik} + \tilde{\gamma}^{-ik})(\tilde{\gamma}^{sk} + \tilde{\gamma}^{-sk})$

(\text{where } \tilde{\gamma} \text{ is a } q - 1 \text{ root of 1 in } \mathbb{C}. \text{)} [3]
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$$\varphi_p(\chi(g)) = \chi(\varphi_p(g))$$
$$= (\tilde{\gamma}^{ikp} + \tilde{\gamma}^{-ikp})(\tilde{\gamma}^{skp} + \tilde{\gamma}^{-skp})$$
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$\varphi_p^m$ is an automorphism of $Sp_4(q)$ that raises all entries of its operand to the power $p^m$. So $\varphi_p^m(\chi(g)) = \chi(\varphi_p^m(g))$.

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$= (\tilde{\gamma}^{ikp} + \tilde{\gamma}^{-ikp})(\tilde{\gamma}^{skp} + \tilde{\gamma}^{-skp})$

So $\varphi_p(\chi_8(k)(g)) = \chi_8(k^p)(g)$.
A $\varphi^m_p$-invariant character is one which can go through $\varphi^m_p$ and come out equal to itself as before the operation.

So if $\chi$ is $\varphi^m_p$-invariant, then $\varphi^m_p(\chi) = \chi$.

For example, if $\chi_8(k)$ is fixed by $\varphi^m_p$, then its values are $\mathbb{Q}$-combinations of $p^m - 1$ roots of unity.
Theorem

Assume every $\varphi^m_p$-invariant member of $\text{Irr}(H)$ is also fixed by $\sigma$.

Then every $\varphi^m_p$-invariant member of $\text{Irr}(q-1)'(G)$ is also fixed by $\sigma$.

Recall $\mathbb{Q}(e^{2\pi i/|G|})$. 

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That is, the rational numbers plus the \(|G|\)th-roots of unity.
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Fun Fact: Although all of our characters do live in \( \mathbb{C} \), we can actually restrict that to \( \mathbb{Q}(e^{2\pi i/|G|}) \).
Theorem
Assume every $\varphi^m_p$-invariant member of $\text{Irr}(H)$ is also fixed by $\sigma$.
Then every $\varphi^m_p$-invariant member of $\text{Irr}_{(q-1)'}(G)$ is also fixed by $\sigma$.

Given an automorphism $\sigma$ of $\mathbb{Q}(e^{2\pi i/|G|})$ and an irreducible character $\chi$ of $G$, we have another irreducible character $(\sigma\chi)$ given by

$$(\sigma\chi)(g) = \sigma(\chi(g))$$
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Given an automorphism \( \sigma \) of \( \mathbb{Q}(e^{2\pi i/|G|}) \) and an irreducible character \( \chi \) of \( G \), we have another irreducible character \( (\sigma\chi) \) given by

\[
(\sigma\chi)(g) = \sigma(\chi(g))
\]

Recall

\[
(\varphi_m \chi)(g) = \chi(\varphi_m(g))
\]

Note that applying \( \sigma \) to \( \chi \) behaves differently than applying \( \varphi_m \) to \( \chi \).
WE MADE IT!

Theorem

Assume every $\varphi_p^m$-invariant member of $\text{Irr}(H)$ is also fixed by $\sigma$. Then every $\varphi_p^m$-invariant member of $\text{Irr}_{(q-1)'}(G)$ is also fixed by $\sigma$. 
The Local Side

**Theorem**

Assume every $\varphi^m_p$-invariant member of $\text{Irr}(H)$ is also fixed by $\sigma$.

Then every $\varphi^m_p$-invariant member of $\text{Irr}(q-1)'(G)$ is also fixed by $\sigma$.

\[ \mathcal{X}_k : \mathbb{F}_q^* \to \mathbb{C}^* \] is an irreducible representation of $\mathbb{F}_q^*$, where
\[ \mathcal{X}_k(\gamma) = \tilde{\gamma}^k, \text{ where } \gamma \text{ is a } q-1 \text{ root of 1 in } \mathbb{F}_q^*. \]

**Lemma**

If $\varphi^m_p$ fixes $\mathcal{X}_k$ then $\tilde{\gamma}^k$ is a $p^m - 1$ root of 1.

(where $\tilde{\gamma}$ is a $q-1$ root of 1 in $\mathbb{C}$.)

All characters of $H$ can be obtained from those of the form $\mathcal{X}_k$. 
The Local Side

Theorem

(*) Assume every $\varphi_p^m$-invariant member of $\text{Irr}(H)$ is also fixed by $\sigma$. Then every $\varphi_p^m$-invariant member of $\text{Irr}(q-1)'(G)$ is also fixed by $\sigma$.

Lemma

Under assumption (*), then every $p^m - 1$ root of 1 is $\sigma$-fixed.
The Global Side

Theorem
Assume every \( \varphi_p^m \)-invariant member of \( \text{Irr}(H) \) is also fixed by \( \sigma \).
Then every \( \varphi_p^m \)-invariant member of \( \text{Irr}(q-1)'(G) \) is also fixed by \( \sigma \).

Assume a character of \( G \) is fixed by \( \varphi_p^m \).
Consider \( \chi_8 \):
Recall that when \( \chi_8(k) \) is fixed by \( \varphi_p^m \), then its values are \( \mathbb{Q} \)-combinations of \( p^m - 1 \) roots of unity.

Lemma
Then its values are in \( \mathbb{Q}(e^{2\pi i/(p^m-1)}) \).
The Global Side

Theorem

(⋆) Assume every \( \varphi^m_p \)-invariant member of \( \text{Irr}(H) \) is also fixed by \( \sigma \).
Then every \( \varphi^m_p \)-invariant member of \( \text{Irr}_{(q-1)'}(G) \) is also fixed by \( \sigma \).

Theorem

Under assumption (⋆), if \( \chi_8(k) \) is fixed by \( \varphi^m_p \) then \( \chi_8(k) \) is also fixed by \( \sigma \).

This, with the previous lemmas, proves our theorem for \( \chi_8(k) \); the proofs for the other members of \( \text{Irr}_{(q-1)'}(G) \) are similar.
Future Direction

Conjecture

Let \( \ell \) be an odd prime and let \( P \) be a Sylow \( \ell \)-subgroup of \( G \) such that \( \varphi_p(g) \in P \) for each \( g \in P \). Let \( m \) be a positive integer and assume every \( \varphi_p^m \)-invariant member of \( \text{Irr}(P) \) is also fixed by \( \sigma_\ell \). Then every \( \varphi_p^m \)-invariant member of \( \text{Irr}_\ell'(G) \) is also fixed by \( \sigma_\ell \). (Here \( \sigma_\ell \) is a specific automorphism of \( \mathbb{Q}(e^{2\pi i/|G|}) \) depending on \( \ell \)).
REFERENCES

Joseph A. Gallian.
*Contemporary Abstract Algebra.*
Houghton Mifflin, Boston, Massachusetts, 2002.

Gordon James and Martin Liebeck.
*Representations and characters of groups.*

Bhama Srinivasan.
The characters of the finite symplectic group $\text{Sp}(4, q)$.
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