Automorphisms and Characters of Finite Groups

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Big Idea

Fixed Notation

- $G = Sp_4(q)$ where q is a power of an odd prime, p
- $H = \{diag(a, a^{-1}, b, b^{-1}) \mid a, b \in F_q\}$ a subgroup of G
- φ_p^m is a "field automorphism" of G
- σ is an automorphism of $\mathbb{Q}(e^{2\pi i/|G|})$

Theorem

Assume every φ_p^m-invariant member of $Irr(H)$ is also fixed by σ. Then every φ_p^m-invariant member of $Irr_{(q-1)'}(G)$ is also fixed by σ.

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$$Sp_4(q) = \{ g \text{ is an invertible } 4 \times 4 \text{ matrix over } \mathbb{F}_q \mid g^T J g = J \}$$

$$\text{where } J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

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- Associativity

  $\forall a, b, c \in G, (a \star b) \star c = a \star (b \star c)$

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  $\forall a, b, c \in G, (a ? b) ? c = a ? (b ? c)$
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  $\exists e \in G$ s.t. $\forall a \in G, a ? e = e ? a = a$.

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Example: $\mathbb{Z}_{12}$ under addition

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Example: $\mathbb{Z}_{12}$ under addition
Non-Example: $\mathbb{Z}_{12}$ under multiplication

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- An inverse for every group element

  Since g^{-1} also satisfies the group definition: $(g^{-1})^T J (g^{-1}) = J$ then every element has an inverse.

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A subgroup H is a subset of group elements of a group G that is itself a group under the group operation.

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Example: The evens mod 12 forms a subgroup of Z_{12} under addition.

Non-Example: The odds mod 12 do not form a subgroup of Z_{12} under addition.

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A diagonal matrix has zeros everywhere except the main diagonal.

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A diagonal matrix has zeros everywhere except the main diagonal.

So $diag(a, a^{-1}, b, b^{-1})$ is the diagonal matrix

$$\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & a^{-1} & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & b^{-1} \end{bmatrix}$$

With entries $a, b \in \mathbb{F}_q$

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But why is H a subgroup of G?

If we let $g = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & a^{-1} & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & b^{-1} \end{bmatrix}$ and $J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$ then

$$g^T J g = \left(\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & a^{-1} & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & b^{-1} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \right) \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & a^{-1} & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & b^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & a & 0 & 0 \\ -a^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & -b^{-1} & 0 \end{bmatrix} \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & a^{-1} & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & b^{-1} \end{bmatrix} = J$$

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But why is H a subgroup of G?

Let $A = \begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_1^{-1} & 0 & 0 \\ 0 & 0 & b_1 & 0 \\ 0 & 0 & 0 & b_1^{-1} \end{bmatrix}$ and $B = \begin{bmatrix} a_2 & 0 & 0 & 0 \\ 0 & a_2^{-1} & 0 & 0 \\ 0 & 0 & b_2 & 0 \\ 0 & 0 & 0 & b_2^{-1} \end{bmatrix}$, then

$$AB = \begin{bmatrix} a_1 a_2 & 0 & 0 & 0 \\ 0 & (a_1 a_2)^{-1} & 0 & 0 \\ 0 & 0 & b_1 b_2 & 0 \\ 0 & 0 & 0 & (b_1 b_2)^{-1} \end{bmatrix}$$

Thus H is closed under the group operation from G, so H is a subgroup of G.

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A homomorphism is a function of one group to another that preserves the group operation.

So for groups $(G, \star)$ and $(G, \cdot)$, then for any $g_1, g_2 \in G$

$$f(g_1 \star g_2) = f(g_1) \cdot f(g_2)$$

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A **homomorphism** is a function of one group to another that preserves the group operation.

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An **automorphism** is a bijective homomorphism from a group G onto itself.

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' $^{m}_{p}$ is a " eld automorphism" of G

is an automorphism of Q(e^{2} $^{i=j\,Gj}$)

' $^{m}_{p}$ is an automorphism of $Sp_4(q)$ that raises all entries of its operand to the power p^m.

Example: let $B_3(i;s)$ 2 G such that $B_3(i;s) =$
$$\begin{bmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & s & 0 \\ 0 & 0 & 0 & s \end{bmatrix}$$

(where is a q 1 root of 1 in F_q)

(as de ned in Srinivasan [3, Srinivasan 1968].)

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Example: let $B_3(i;s) \in G$ such that
$$B_3(i;s) = \begin{bmatrix} \omega^{i} & 0 & 0 & 0 \\ 0 & \omega^{-i} & 0 & 0 \\ 0 & 0 & \omega^{s} & 0 \\ 0 & 0 & 0 & \omega^{-s} \end{bmatrix}$$

(where ω is a $q-1$ root of 1 in $\mathbb{F}_q$)

(as defined in Srinivasan [3, Srinivasan 1968].)

Observe $\varphi_p(B_3(i;s))$:
$$\varphi_p\left(\begin{bmatrix} \omega^{i} & 0 & 0 & 0 \\ 0 & \omega^{-i} & 0 & 0 \\ 0 & 0 & \omega^{s} & 0 \\ 0 & 0 & 0 & \omega^{-s} \end{bmatrix}\right) = \begin{bmatrix} \omega^{ip} & 0 & 0 & 0 \\ 0 & \omega^{-ip} & 0 & 0 \\ 0 & 0 & \omega^{sp} & 0 \\ 0 & 0 & 0 & \omega^{-sp} \end{bmatrix} = B_3(ip; sp)$$

Representation

A *representation* is a homomorphism ρ from a group G into a group of $n \times n$ invertible matrices with entries in $\mathbb{C}$.

$$\rho : G \to GL_n(\mathbb{C}) \text{ such that}$$
$$\rho(gh) = \rho(g)\rho(h) \text{ for all } g, h \in G$$

(where $GL_n(\mathbb{C})$ is the group of $n \times n$ invertible matrices with entries in $\mathbb{C}$)

Trace

The traceof a matrix is the sum of its diagonal entries.

So $Tr(A_{n \times n}) = a_{11} + a_{22} + \cdots + a_{nn}$.

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So if $h = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & a^{-1} & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & b^{-1} \end{bmatrix}$

Then $Tr(h) = a + a^{-1} + b + b^{-1}$.

Character

A character is the composition of the trace function with the representation of a group element.

$$= \mathrm{Tr}$$

$$(g) = \mathrm{Tr}(\quad (g))$$

RECALL ...

Theorem

Assume every ${}_p^m$-invariant member of $\mathrm{Irr}(H)$ is also xed by .
Then every ${}_p^m$-invariant member of $\mathrm{Irr}_{(q\quad 1)^0}(G)$ is also xed by .

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is irreducibleif ≠ $_1$ + $_2$ for characters $_1$; $_2$

Irr (H) is the set of irreducible characters of H.

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is irreducibleif $\neq$ $_1$ + $_2$ for characters $_1$; $_2$

Irr (H) is the set of irreducible characters of H.

Irr $_{(q\ \ 1)^0}$(G) is the set of irreducible characters of G such that n is relatively prime to the quantity (q 1).

Theorem

Assume every ' $_p^m$-invariant member of Irr (H) is also xed by .
Then every ' $_p^m$-invariant member of Irr $_{(q\ \ 1)}$ 0(G) is also xed by .

Given the automorphism of ' $_p^m$ of G and 2 Irr (G), we can obtain a new irreducible character ' $_p^m$ via

$$' {}_p^m (g) = (' {}_p^m(g))$$

Theorem

Assume every $'^{m}_{p}$-invariant member of Irr (H) is also xed by .
Then every $'^{m}_{p}$-invariant member of Irr $_{(q\ \ 1)}0(G)$ is also xed by .

$'^{m}_{p}$ is an automorphism of $Sp_4(q)$ that raises all entries of its operand to the power p^m. So $'^{m}_{p}(\ (g)) = \ ('^{m}_{p}(g))$.

Looking at $_8(k)$, we claim that $'_p(\ _8(k)) = \ _8(kp)$.

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Looking at $_8$(k), we claim that ' $_p$($_8$(k)) = $_8$(kp).

For example, we know that ' $_p(B_3(i;s)) = B_3(ip;sp)$.
Now let = $_8$(k) and g = $B_3(i;s)$, where $_8$ is a character of G.

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Now let $= \ _8(k)$ and $g = B_3(i;s)$, where $_8$ is a character of G.

Notice that $(g) = (\sim^{ik} + \sim^{ik})(\sim^{sk} + \sim^{sk})$

(where $\sim$ is a $q\ \ 1$ root of 1 in $\mathbb{C}$.) [3]

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φ_p^m is an automorphism of $Sp_4(q)$ that raises all entries of its operand to the power p^m. So $\varphi_p^m(\ (g)) = \ (\varphi_p^m(g))$.

Looking at $_8(k)$, we claim that $\varphi_p(\ _8(k)) = \ _8(kp)$.

For example, we know that $\varphi_p(B_3(i;s)) = B_3(ip;sp)$.
Now let $= \ _8(k)$ and $g = B_3(i;s)$, where $_8$ is a character of G.

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(where $\sim$ is a $q\ \ 1$ root of 1 in $\mathbb{C}$.) [3]

Consider $\varphi_p(\ (g)) = \ (\varphi_p(g))$
$= (\sim^{ikp} + \sim\ ^{ikp})(\sim^{skp} + \sim\ ^{skp})$

So $\varphi_p(\ _8(k)(g)) = \ _8(kp)(g)$.

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Then every ' $_p^m$-invariant member of Irr $_{(q\ \ 1)}$ $_0$(G) is also xed by .

A ' $_p^m$-invariant character is one which can go through ' $_p^m$ and come out equal to itself as before the operation.

So if is ' $_p^m$-invariant, then ' $_p^m$( ) = .

For example, if $_8$(k) is xed by ' $_p^m$, then its values are Q-combinations of p^m 1 roots of unity.

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Then every ' $^{m}_{p}$-invariant member of Irr $_{(q \quad 1)}{}_{0}(G)$ is also xed by .

Recall Q(e$^{2 \quad i=jGj}$).

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That is, the rational numbers plus the jGjth-roots of unity.

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That is, the rational numbers plus the jGjth-roots of unity.

Fun Fact: Although all of our characters do live in C, we can actually restrict that to $Q(e^{2 \quad i=jGj})$.

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Given an automorphism of Q(e$^{2 \quad i=jGj}$) and an irreducible character of G,
we have another irreducible character ( ) given by

$$(\quad)(g) = \quad (\quad (g))$$

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Assume every φ_p^m-invariant member of Irr (H) is also xed by .
Then every φ_p^m-invariant member of Irr $_{(q \quad 1)'}(G)$ is also xed by .

Given an automorphism of $\mathbb{Q}(e^{2 \quad i=jGj})$ and an irreducible character of G,
we have another irreducible character ( ) given by

$$(\quad)(g) = \quad (\quad (g))$$

Recall

$$(\varphi_p^m)(g) = \quad (\varphi_p^m(g))$$

Note that applying to behaves differently than applying φ_p^m to .

We made it !

Theorem

Assume every $_{p}^{m}$-invariant member of Irr (H) is also xed by .
Then every $_{p}^{m}$-invariant member of Irr$_{(q \;\; 1)^{0}}$(G) is also xed by .

THE LOCAL SIDE

Theorem

Assume every ' $_p^m$-invariant member of Irr (H) is also xed by .

Then every ' $_p^m$-invariant member of Irr $_{(q\ \ 1)}$0(G) is also xed by .

$X_k : F_q$! C is an irreducible representation of F_q, where $X_k(\ ) = \sim^k$, where is a q 1 root of 1 in F_q.

Lemma

If ' $_p^m$ xes X_k then $\sim^k$ is a p^m 1 root of 1.

(where ~ is a q 1 root of 1 in C.)

All characters of H can be obtained from those of the form X_k.

THE LOCAL SIDE

Theorem

(?) Assume every ' $^{m}_{p}$-invariant member of Irr (H) is also xed by .
Then every ' $^{m}_{p}$-invariant member of Irr $_{(q\ \ 1)}$ 0(G) is also xed by .

Lemma

Under assumption (?), then every p^{m} 1 root of 1 is - xed.

THE GLOBAL SIDE

Theorem

Assume every ' $^{m}_{p}$-invariant member of Irr (H) is also xed by .

Then every ' $^{m}_{p}$-invariant member of Irr $_{(q\ \ 1)}$ 0 (G) is also xed by .

Assume a character of G is xed by ' $^{m}_{p}$.

Consider $_8$:

Recall that when $_8$(k) is xed by ' $^{m}_{p}$, then its values are Q-combinations of p^m 1 roots of unity.

Lemma

Then its values are in $\mathbb{Q}(e^{2\ \ i=(p^m\ \ 1)})$.

The Global Side

Theorem

(?) Assume every ' $^{m}_{p}$-invariant member of Irr (H) is also xed by .

Then every ' $^{m}_{p}$-invariant member of Irr $_{(q\ \ 1)^{0}}$(G) is also xed by .

Theorem

Under assumption (?), if $_{8}$(k) is xed by ' $^{m}_{p}$ then $_{8}$(k) is also xed by .

This, with the previous lemmas, proves our theorem for $_{8}$(k); the proofs for the other members of Irr $_{(q\ \ 1)^{0}}$(G) are similar.

Future Direction

Conjecture

Let ℓ be an odd prime and let P be a Sylow ℓ-subgroup of G such that $\varphi_p(g) \in P$ for each $g \in P$. Let m be a positive integer and assume every φ_p^m-invariant member of $\mathrm{Irr}(P)$ is also fixed by σ. Then every φ_p^m-invariant member of $\mathrm{Irr}_{\ell'}(G)$ is also fixed by σ. (Here σ is a specific automorphism of $\mathbb{Q}(e^{2\pi i/|G|})$ depending on ℓ.)

References

Joseph A. Gallian.
Contemporary Abstract Algebra.
Houghton Mifflin, Boston, Massachusetts, 2002.

Gordon James and Martin Liebeck.
Representations and characters of groups.
Cambridge University Press, New York, second edition, 2001.

Bhama Srinivasan.
The characters of the finite symplectic group $Sp(4, q)$.
Trans. Amer. Math. Soc., 131:488–525, 1968.