Efficiency of a Moving Mesh System with a Curvature-type Monitor Applied to Burgers' Equation

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Outline

1. Burgers' Equation
2. Physical Solution PDE & Errors
3. Moving Mesh PDE & Benefits
4. Our Theorem
5. Why it matters

An Interesting RDM: Burgers' Equation

- Simplified Navier-Stokes equation, in 1-D:

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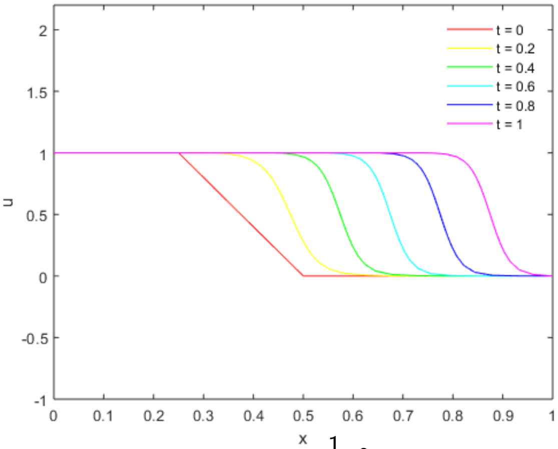
- Propagating wavefront with steepness controlled by ϵ

Evolution of a Numerical Solution to Burgers' Equation Over Time ($\epsilon = 0.01$)



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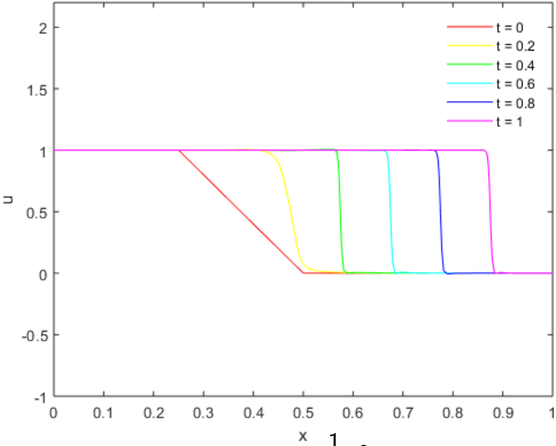

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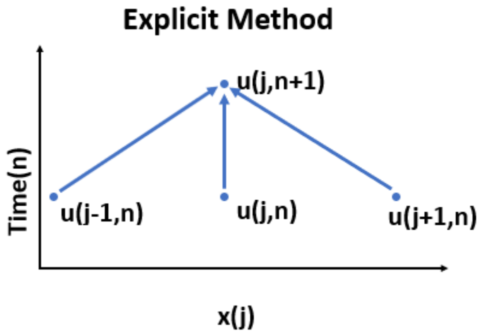
Approximating Solutions over Time

- Finding $u(x_j, t_{n+1})$:

$$u_{j,n+1} = \left(\frac{\epsilon \Delta t}{h_j^2}\right) u_{j-1,n} + \left(1 - 2\frac{\epsilon \Delta t}{h_j^2}\right) u_{j,n} + \left(\frac{\epsilon \Delta t}{h_j^2}\right) u_{j+1,n} + \frac{\Delta t}{4h_j}(u_{j+1,n}^2 - u_{j-1,n}^2) + u_{j,n}$$



Introduction to Moving Mesh Methods

- Adaptive techniques to solve partial differential equations numerically
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Goal

Balance the undesirable characteristics of the physical PDE by adjusting points using a moving mesh PDE.

The Moving Mesh Equation

- Moving Mesh PDE:

$$x_t = (\omega x_\xi)_\xi$$

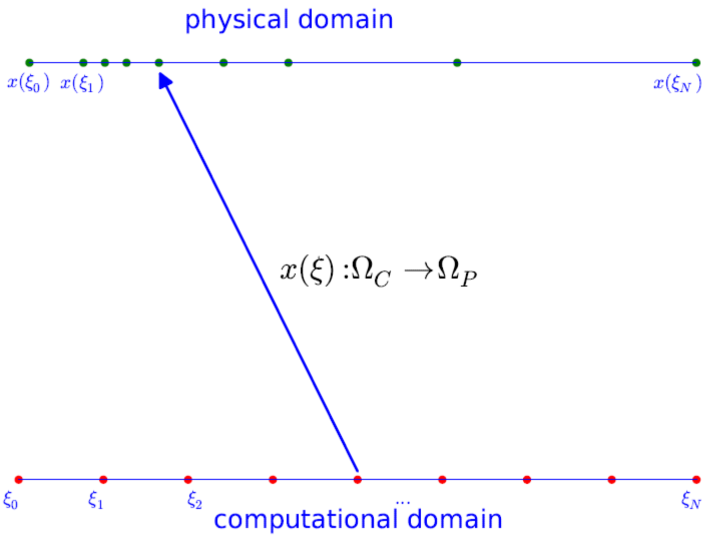
for $x = x(\xi, t)$

- Steady State Moving Mesh PDE:

$$0 = (\omega x_\xi)_\xi$$

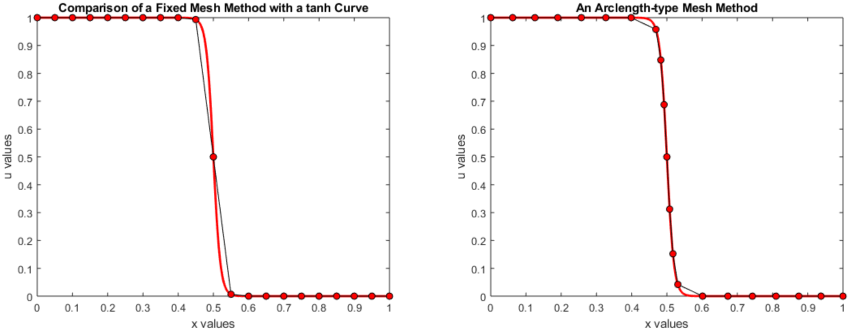
- $\omega =$ Monitor Function,
aka the "Mesh Density Function"

Mesh Movement Mapping



Examples of Moving Mesh

Figure: A fixed mesh method compared to an Arc Length-type mesh



$$\omega = \sqrt{(1 + \alpha u_x^2)}$$

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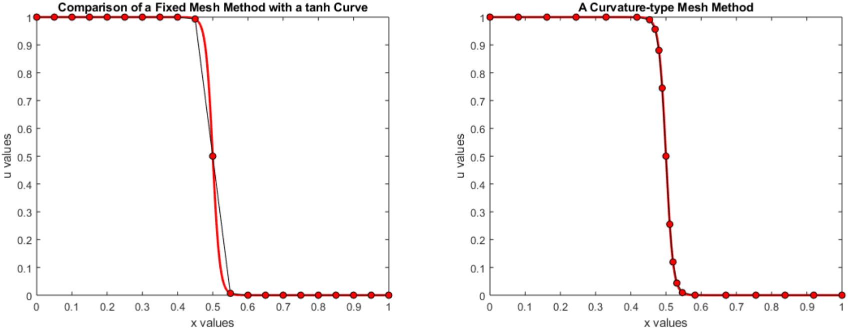
Figure: A fixed mesh method compared to a Curvature-type mesh



$$\omega = (1 + \epsilon^p u_{xx}^2)^{1/q}$$

Effectiveness of the Curvature Monitor

Here, note that for z to be $O(C)$ means that $M_1 C \leq z \leq M_2 C$, where M_1 and M_2 are arbitrary constants.

Theorem (DKRY'18)

Let $u = u(x)$ be the physical solution that satisfies the following assumptions:
(i) the solution has large gradient in Ω_ϵ, i.e., $\|u_x\|_\infty = O(\epsilon^{-1})$ and in $[0,1] \setminus \Omega_\epsilon$ $\|u_x\|_\infty = O(1)$, and
(ii) the solution has large curvature over Ω_ϵ, i.e., $\|u_{xx}\|_\infty = O(\epsilon^{-2})$ and in $[0,1] \setminus \Omega_\epsilon$ $\|u_{xx}\|_\infty = O(1)$, where $\text{meas}(\Omega_\epsilon) = O(\epsilon)$.

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where $\epsilon \leq 1$, p and q are nonnegative and $p + q \geq 4$, the solution in computational domain, $v(\xi) = u(x(\xi))$, and the mapping from the physical domain to the computational domain, $\xi = \xi(x)$, satisfy the following bounds:

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$$\|x_\xi\|_\infty = O(1), \quad \|\xi_x\|_\infty = O(\epsilon^{\frac{p-4}{q}}),$$
$$\text{and} \quad 0 \le \|v_\xi\|_\infty \le M\epsilon^{\frac{4-p-q}{q}}.$$

Corollary When Considering a Discrete System

Corollary (DKRY'18)

When considering the system discretely, with the same hypotheses as previously, where $h_j = x_{j+1} - x_j$, *the following bounds are satisfied:*
(i) On $[0,1] \setminus \Omega_\epsilon \quad \min h_j = O(\Delta\xi)$
(ii) On Ω_ϵ*:* $\min h_j = O(\epsilon^{\frac{4-p}{q}} \Delta\xi)$

Mistakes Were Made: Types of Errors ☹

- Truncation error:

$$u_x(x_j) = \frac{u(x_{j+1}) - u(x_{j-1})}{2h_j} + (2h_j)^2 u_{xx}(x_j) + \ldots$$

- When u_{xx} is large (we assume $O(\epsilon^{-2})$), we need h_j very small
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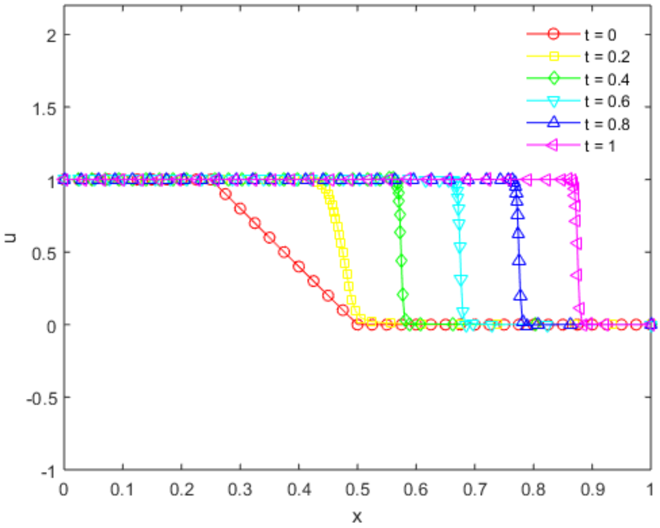
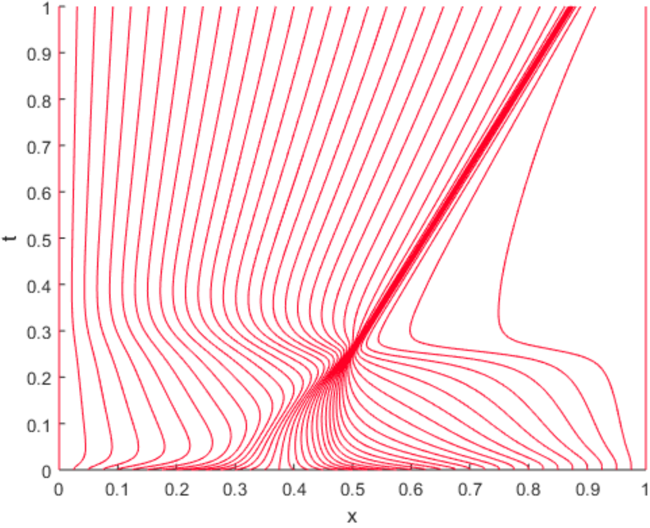
- A moving mesh uses $\min h_j = O(\Delta\xi\, \epsilon^{\frac{4-p}{q}})$ . . . $p = 1, q = 6$
- On a fixed mesh, the truncation error is of order $\Delta\xi^2\epsilon^{-2}$, but on this moving mesh system, truncation error is of order $\Delta\xi^2\epsilon^{-1}$

Example of Moving Mesh on Burgers' Equation



Mesh Trajectories for the Modeled Solution



Numerical Evidence for an Approximated Solution of Burgers' Equation

Table: ϵ values for $\omega = (1 + \epsilon u_{xx}^2)^{1/6}$

ϵ	$\|\|u_x\|\|_\infty$	$\|\|v_\xi\|\|_\infty$	exp	min h_j	exp	$\|\|\xi_x\|\|_\infty$	exp
0.01	12.449	3.193	-0.448	0.00319	0.713	5.220	-0.713
0.005	24.919	4.357	-0.451	0.00195	0.665	8.556	-0.658
0.0025	49.495	5.958	-0.452	0.00123	0.611	13.498	-0.615
0.00125	97.907	8.152	-0.451	0.00080	0.568	20.672	-0.567
0.000625	193.055	11.145		0.00054		30.606	

$$0 \le ||v_\xi||_\infty \le M\epsilon^{\frac{-1}{2}}, \ \ \min h_j = O(\epsilon^{\frac{1}{2}}) \quad \textit{and} \quad ||\xi_x||_\infty = O(\epsilon^{\frac{-1}{2}}).$$

Any Questions? ¯_☺_/¯

Thank You!

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