Efficiency of a Moving Mesh System with a Curvature-type Monitor Applied to Burgers’ Equation

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Outline

1. Burgers’ Equation
2. Physical Solution PDE & Errors
3. Moving Mesh PDE & Benefits
4. Our Theorem
5. Why it matters
An Interesting RDM: Burgers’ Equation

- Simplified Navier-Stokes equation, in 1-D:

\[ u_t = \epsilon u_{xx} - \left( \frac{1}{2} u^2 \right)_x \]
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Initial conditions:

\[ u(x, 0) = \begin{cases} 
1 & x \leq 0.25 \\
2 - 4x & 0.25 < x \leq 0.5 \\
0 & x > 0.5 
\end{cases} \]
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- Propagating wavefront with steepness controlled by \( \epsilon \)
Evolution of a Numerical Solution to Burgers’ Equation Over Time ($\epsilon = 0.01$)

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Approximating Solutions over Time

- Finding \( u(x_j, t_{n+1}) \):

\[
u_{j,n+1} = \left( \frac{\epsilon \Delta t}{h_j^2} \right) u_{j-1,n} + \left( 1 - 2 \frac{\epsilon \Delta t}{h_j^2} \right) u_{j,n} + \left( \frac{\epsilon \Delta t}{h_j^2} \right) u_{j+1,n} + \frac{\Delta t}{4h_j} (u_{j+1,n} - u_{j-1,n}) + u_{j,n}\]
Introduction to Moving Mesh Methods

- Adaptive techniques to solve partial differential equations numerically
- As physical solution, $u$, evolves, so do the grid points, $x_j$
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Goal
Balance the undesirable characteristics of the physical PDE by adjusting points using a moving mesh PDE.
The Moving Mesh Equation

- **Moving Mesh PDE:**
  \[ x_t = (\omega x_\xi)_\xi \]
  for \( x = x(\xi, t) \)

- **Steady State Moving Mesh PDE:**
  \[ 0 = (\omega x_\xi)_\xi \]

- \( \omega = \text{Monitor Function}, \) aka the “Mesh Density Function”
Mesh Movement Mapping

\[ x(\xi) : \Omega_C \rightarrow \Omega_P \]
Examples of Moving Mesh

Figure: A fixed mesh method compared to an Arc Length-type mesh

\[ \omega = \sqrt{1 + \alpha u_x^2} \]
Moving Mesh Methods

Examples of Moving Mesh

Figure: A fixed mesh method compared to a Curvature-type mesh

\[
\omega = \left(1 + \epsilon^p u_{xx}^2\right)^{1/q}
\]
Effectiveness of the Curvature Monitor

Here, note that for $z$ to be $O(C)$ means that $M_1 C \leq z \leq M_2 C$, where $M_1$ and $M_2$ are arbitrary constants.

**Theorem (DKRY’18)**

Let $u = u(x)$ be the physical solution that satisfies the following assumptions:

(i) the solution has large gradient in $\Omega_\epsilon$, i.e., $\|u_x\|_\infty = O(\epsilon^{-1})$ and in $[0, 1] \div \Omega_\epsilon$ $\|u_x\|_\infty = O(1)$, and

(ii) the solution has large curvature over $\Omega_\epsilon$, i.e., $\|u_{xx}\|_\infty = O(\epsilon^{-2})$ and in $[0, 1] \div \Omega_\epsilon$ $\|u_{xx}\|_\infty = O(1)$, where $\text{meas}(\Omega_\epsilon) = O(\epsilon)$. 

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(i) the solution has large gradient in $\Omega_\epsilon$, i.e., $\|u_x\|_\infty = O(\epsilon^{-1})$ and in $[0,1] \setminus \Omega_\epsilon$ $\|u_x\|_\infty = O(1)$, and
(ii) the solution has large curvature over $\Omega_\epsilon$, i.e., $\|u_{xx}\|_\infty = O(\epsilon^{-2})$ and in $[0,1] \setminus \Omega_\epsilon$ $\|u_{xx}\|_\infty = O(1)$, where $\text{meas}(\Omega_\epsilon) = O(\epsilon)$. Then, with the monitor function

$$\omega = (1 + \epsilon^p u_{xx}^2)^{1/q},$$

where $\epsilon \leq 1$, $p$ and $q$ are nonnegative and $p + q \geq 4$, the solution in computational domain, $v(\xi) = u(x(\xi))$, and the mapping from the physical domain to the computational domain, $\xi = \xi(x)$, satisfy the following bounds:
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$$\|x_\xi\|_\infty = O(1), \quad \|\xi_x\|_\infty = O(\varepsilon^{\frac{p-4}{q}}),$$

and $0 \leq \|v_\xi\|_\infty \leq M \varepsilon^{\frac{4-p-q}{q}}$. 
Corollary (DKRY’18)

When considering the system discretely, with the same hypotheses as previously, where $h_j = x_{j+1} - x_j$, the following bounds are satisfied:

(i) On $[0, 1] \setminus \Omega_{\epsilon}$: \[ \min h_j = O(\Delta \xi) \]

(ii) On $\Omega_{\epsilon}$: \[ \min h_j = O(\epsilon^{\frac{4-p}{q}} \Delta \xi) \]
Mistakes Were Made: Types of Errors 😞

- **Truncation error:**

  \[ u_x(x_j) = \frac{u(x_{j+1}) - u(x_{j-1})}{2h_j} + (2h_j)^2 u_{xx}(x_j) + ... \]

- When \( u_{xx} \) is large (we assume \( O(\epsilon^{-2}) \)), we need \( h_j \) very small
- A fixed mesh uses \( h_j = \Delta \xi \)
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- A moving mesh uses \( \min h_j = O(\Delta \xi \epsilon^{\frac{4-p}{q}}) \) . . . \( p = 1, q = 6 \)

- On a fixed mesh, the truncation error is of order \( \Delta \xi^2 \epsilon^{-2} \), but on this moving mesh system, truncation error is of order \( \Delta \xi^2 \epsilon^{-1} \)
Example of Moving Mesh on Burgers’ Equation
Mesh Trajectories for the Modeled Solution
Numerical Evidence for an Approximated Solution of Burgers’ Equation

Table: $\epsilon$ values for $\omega = (1 + \epsilon u_{xx}^2)^{1/6}$

| $\epsilon$ | $||u_x||_\infty$ | $||v_\xi||_\infty$ | exp | min $h_j$ | exp | $||\xi_x||_\infty$ | exp |
|------------|-----------------|-----------------|-----|-----------|-----|----------------|-----|
| 0.01       | 12.449          | 3.193           | -0.448 | 0.00319 | 0.713 | 5.220          | -0.713 |
| 0.005      | 24.919          | 4.357           | -0.451 | 0.00195 | 0.665 | 8.556          | -0.658 |
| 0.0025     | 49.495          | 5.958           | -0.452 | 0.00123 | 0.611 | 13.498         | -0.615 |
| 0.00125    | 97.907          | 8.152           | -0.451 | 0.00080 | 0.568 | 20.672         | -0.567 |
| 0.000625   | 193.055         | 11.145          | -0.451 | 0.00054 |       | 30.606         |       |

$0 \leq ||v_\xi||_\infty \leq M\epsilon^{-\frac{1}{2}}, \text{ min } h_j = O(\epsilon^{\frac{1}{2}}) \text{ and } ||\xi_x||_\infty = O(\epsilon^{-\frac{1}{2}})$. 
Thank You!

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