Braces and their Opposites

Sink Your Teeth Into Algebra!

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Motivation

The Yang-Baxter Equation

- Important equation in theoretical physics, specifically with applications in quantum computing

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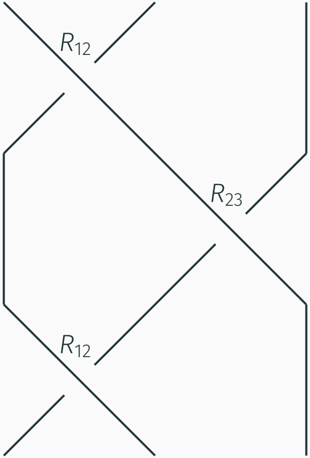
Braid Relation



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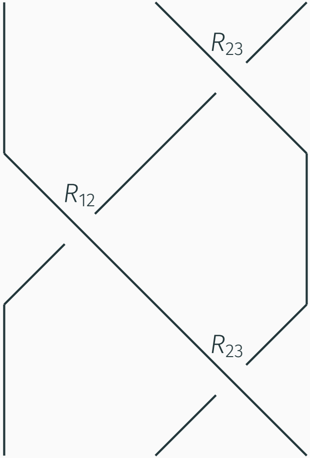

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- $(a * b) * c = a * (b * c)$;
- there is some $e \in G$ with $a * e = e * a = a$;
- there is some $a^{-1} \in G$ with $a * a^{-1} = a^{-1} * a = e$.

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Right skew braces are defined analogously, but not used to find set-theoretic solutions to the Yang-Baxter equation.

Examples

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- $(B, \cdot, \circ)$, where $a \cdot b = (a + b) \bmod 2p$ and $a \circ b = a + (-1)^a b$.

$$\begin{aligned} a \circ (bc) &= a + (-1)^a(b + c) \\ &= a + (-1)^a b + (-1)^a c \\ &= a + (-1)^a b - a + a + (-1)^a c \\ &= (a \circ b)a^{-1}(a \circ c) \end{aligned}$$

The Yang-Baxter Equation Revisited

Let $(B, \cdot, \circ)$ be a brace.
Then $R : B \times B \to B \times B$ is defined by

$$R(a, b) = (a^{-1}(a \circ b), \overline{a^{-1}(a \circ b)} \circ a \circ b),$$

where $a \circ \overline{a} = \overline{a} \circ a = e$.

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where $a \circ \overline{a} = \overline{a} \circ a = e$.
For example, a trivial brace gives us

$$\begin{aligned} R(a,b) &= (a^{-1}(a \circ b), \overline{a^{-1}(a \circ b)} \circ a \circ b) \\ &= (a^{-1}ab, (a^{-1}ab)^{-1}ab) \\ &= (b, b^{-1}ab). \end{aligned}$$

Truman Opposite

Definition

The Truman Opposite of a brace $(B, \cdot, \circ)$ is defined to be the brace $(B, \cdot, \circ')$, where $a \circ' b = (a^{-1} \circ b^{-1})^{-1}$. We can show that this satisfies the brace condition:

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$$\begin{aligned} a \circ' (bc) &= (a^{-1} \circ (bc)^{-1})^{-1} \\ &= (a^{-1} \circ c^{-1}b^{-1})^{-1} \\ &= ((a^{-1} \circ c^{-1})a(a^{-1} \circ b^{-1}))^{-1} \\ &= (a^{-1} \circ b^{-1})^{-1}a^{-1}(a^{-1} \circ c^{-1})^{-1} \\ &= (a \circ' b)a^{-1}(a \circ' c). \end{aligned}$$

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- $(B, \cdot, \circ)$ such that $x \circ y = \begin{cases} xy \text{ if } x \text{ or } y \in \langle r \rangle \\ r^2xy \text{ if } x, y \notin \langle r \rangle \end{cases}$
- $B' = (B, \cdot, \circ')$ such that $x \circ' y = \begin{cases} yx \text{ if } x \text{ or } y \in \langle r \rangle \\ r^2yx \text{ if } x, y \notin \langle r \rangle \end{cases}$

Conjecture

$B \cong B'$ if and only if $(B, \cdot)$ is abelian.

A bijective function $\phi : A \to B$ is a brace isomorphism if $\phi(xy) = \phi(x)\phi(y)$ and $\phi(x \circ y) = \phi(x) \circ \phi(y)$.

Proposition (G-S)

Let $(B, \cdot, \circ)$ be a left skew brace where there exists an $a, b \in B$ such that $a \circ b = ab$, and $ab \neq ba$. Then $B \ncong B'$.

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Proof.

Assume there exists a brace isomorphism $\phi : B \to B'$.
Then

$$\begin{aligned}
\phi(ab) &= \phi(a \circ b) \\
&= \phi(a) \circ' \phi(b) \\
&= (\phi(a)^{-1} \circ \phi(b)^{-1})^{-1} \\
&= (\phi(a)^{-1}\phi(b)^{-1})^{-1} \\
&= \phi(b)\phi(a) = \phi(ba).
\end{aligned}$$

But $ab \neq ba$ so ϕ cannot be injective and is not an isomorphism.
Therefore $B \ncong B'$. □

The Yang-Baxter Equation Re-revisited

Let R' denote the set-theoretical solution to the Yang-Baxter given by B'. We conjecture that $R' = TRT$, where $T(a, b) = (b, a)$.

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This is immediately true for the trivial brace, where $R(a, b) = (b, b^{-1}ab)$ and $R'(a, b) = (a^{-1}ba, a)$:

$$\begin{aligned} T(R(T(a, b))) &= T(R(b, a)) \\ &= T(a, a^{-1}ba) \\ &= (a^{-1}ba, a). \end{aligned}$$

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Questions?

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