Modular Forms and Elliptic Curves

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Definitions

Let τ be an element of the upper half plane, $\mathbb{H}$, and let $q := e^{2\pi i\tau}$.

- The Dedekind η-function is defined as:

$$\eta(\tau) := q^{1/24} \prod_{n=1} (1 - q^n).$$

- Let $f(\tau)$ be a meromorphic function on $\mathbb{H}$ of the form:

$$f(\tau) = \prod_{0<\delta|N} \eta(\delta\tau)^{r_\delta}$$

We call f an η-quotient. If all of the r_δ are non-negative integers, then we call f an η-product.

Motivation

The η-function and η-quotients have arisen in many important works of mathematics, including:

- (1918) Ramanujan's work on the partition generating function, $p(n)$:

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{m=1}^{\infty} \left(\frac{1}{1-q^m} \right) = q^{-\frac{1}{24}} \eta(\tau).$$

- (1992) Conway, Norton, and Borcherd's work on monstrous moonshine.

Moreover,

- (1997) Martin and Ono listed all weight 2 newforms that are products and quotients of the η-function.
- (2012) Pathakjee, RosnBrick, and Yoong provide explicit representations of modular forms associated to certain elliptic curves as linear combinations of η-quotients.

  An elliptic curve over $\mathbb{Q}$, denoted $E/\mathbb{Q}$, is given by an equation of the form

$$E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6, \ \textit{where } a_i \in \mathbb{Q}.$$

Recent Work

In addition,

- (2014) Rouse and Webb studied the spaces of modular forms spanned by η-quotients.
- (2016) Arnold-Roksandich, James, and Keaton considered η-quotients of prime level.

Motivating Questions

1. Can we push the method of Pathakjee, RosnBrick, and Yoong to get additional results?

2. What more can we say about generating spaces of modular forms by η-quotients? What about when we restrict to prime or semi-prime level?

The Modular Group and Congruence Subgroups

We denote modular group $SL_2(\mathbb{Z})$ the set of all 2×2 matrices

$$\left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mid a, b, c,\ d \in \mathbb{Z},\ ad - bc = 1 \right\}$$

We are particularly interested in the subgroups of $SL_2(\mathbb{Z})$ which contain the identity matrix modulo N, in particular, we define:

$$\Gamma_0(N) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}) \mid c \equiv 0 \mod N \right\}$$
$$\Gamma_1(N) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}) \mid c \equiv 0 \mod N,\ a \equiv d \equiv 1 \mod N \right\}$$
$$\Gamma(N) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z}) \mid b \equiv c \equiv 0 \mod N,\ a \equiv d \equiv 1 \mod N \right\}$$

Our research is concerned with modular forms

- Let $f : \mathbb{H}^* \to \mathbb{C}$, where $H^* := \mathbb{H} \cup \mathbb{Q} \cup \{i\infty\}$, and $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_N$
- If f is holomorphic on H^* we call it a modular form, denoted $M_k(\Gamma_N, \chi)$.
- If f is meromorphic on H^* we call it a weakly holomorphic modular form, denoted $M_k^!(\Gamma_N, \chi)$.
- If f is a modular form which has positive integral orders of vanishing at the cusps of Γ_N, we call f a cusp form, denoted $S_k(\Gamma_N, \chi)$.

A Useful Theorem

The following theorem of Gordon and Hughes is commonly used in the study of modular η-quotients.

Theorem 1 ([6])

If η_g is an η-quotient given by $\eta_g = \prod_{0<\delta|N} \eta^{r_\delta}(\delta\tau)$, and if

$$\sum_{0<\delta|N} \delta r_\delta \equiv 0 \mod 24 \quad (1)$$

$$\sum_{0<\delta|N} \frac{N}{\delta} r_\delta \equiv 0 \mod 24 \quad (2)$$

then $\eta_g(\tau) \in M_k^!(\Gamma_0(N), \chi)$, where $k = \frac{1}{2}\sum_{0<\delta|N} r_\delta$, $\chi = \left(\frac{(-1)^k s}{d}\right)$, $s = \prod_{0<\delta|N} \delta^{r_\delta}$.

Another Useful Theorem

Gordon and Hughes also prove the following theorem

Theorem 2 ([6])

Let c, d, and N be positive integers with $d \mid N$ *and* $\gcd(c, d) = 1$*. If* $f(\tau)$ *is an* η*-quotient satisfying conditions 1 and 2 given in Theorem 1 for N then the order of vanishing for* $f(\tau)$ *at the cusp* $\frac{c}{d}$ *of any level N congurnece subgroup* Γ_N *is:*

$$\frac{N}{24} \sum_{0<\delta|N} \frac{\gcd(d,\delta)^2 r_\delta}{\gcd(d,\frac{N}{d}) d\delta}$$

Modularity Theorem

Every elliptic curve $E/\mathbb{Q}$ with conductor N has an L-function,

$$L(E;s) = \sum_{n=1}^{\infty} \frac{a_E(n)}{n^s}$$

such that the Fourier series $f : \mathbb{H} \to \mathbb{C}$ given by

$$f(\tau) = \sum_{n=1}^{\infty} a_E(n) q^n; \qquad q = e^{2\pi i \tau}$$

represents a weight 2 newform of level N.

Nice representations of modular forms associated to elliptic curves

Consider the following elliptic curves E_N of conductor N:

- $E_{33} : y^2 = x(x-1)(x - \frac{27}{16})$
- $E_{40} : y^2 = x(x-1)(x-5)$
- $E_{42} : y^2 = x(x-1)(x - \frac{81}{49})$
- $E_{70} : y^2 = x(x-1)(x - (-\frac{7}{25}))$

Nice representations of modular forms associated to elliptic curves

Theorem 3 (Pathakjee, RosnBrick, Yoong, 2012)

The elliptic curves E_N correspond to the following modular forms:

- $E_{33} \leftrightarrow$ $\eta(\tau)^2\eta(11\tau)^2 + 3[\eta(3\tau)^2\eta(33\tau)^2] + 3[\eta(\tau)\eta(3\tau)\eta(11\tau)\eta(33\tau)]$
- $E_{40} \leftrightarrow \eta(\tau)^{-1}\eta(2\tau)^2\eta(4\tau)^2\eta(5\tau)\eta(8\tau)^{-1}\eta(40\tau) + \eta(\tau)\eta(5\tau)^{-1}\eta(8\tau)\eta(10\tau)^2\eta(20\tau)^2\eta(40\tau)^{-1}$
- $E_{42} \leftrightarrow 2[\eta(\tau)^{-1}\eta(2\tau)^2\eta(3\tau)\eta(7\tau)^2\eta(14\tau)^{-1}\eta(42\tau)] - 3[\eta(3\tau)\eta(6\tau)\eta(21\tau)\eta(42\tau)] + \eta(2\tau)\eta(3\tau)^2\eta(6\tau)^{-1}\eta(7\tau)\eta(21\tau)^{-1}\eta(42\tau)^2 + \eta(\tau)\eta(3\tau)^{-1}\eta(6\tau)^2\eta(14\tau)\eta(21\tau)^2\eta(42\tau)^{-1}$
- $E_{70} \leftrightarrow$ $\eta(\tau)^{-1}\eta(2\tau)^2\eta(5\tau)^2\eta(7\tau)^{-1}\eta(10\tau)^{-1}\eta(14\tau)^2\eta(35\tau)^2\eta(70\tau)^{-1} - \eta(\tau)^2\eta(2\tau)^{-1}\eta(5\tau)^{-1}\eta(7\tau)^2\eta(10\tau)^2\eta(14\tau)^{-1}\eta(35\tau)^{-1}\eta(70\tau)^2$

Method

For our work, we wanted to find additional examples of this type using some of our results on η-quotients of semi-prime level.

We generate linear combinations of η-quotients by the following steps:

- Choose a level N, such that N is a semiprime that is coprime to 6.
- Compute $\dim_{\mathbb{C}} S_2(\Gamma_0(N))$.

We are specifically looking at η-quotients of the form

$$\eta(\tau)^{r_1}\eta(p\tau)^{r_p}\eta(q\tau)^{r_q}\eta(N\tau)^{r_N},$$

where p, q are the primes that divide the level N.

Order of Vanishings

Corollary 4 (A.H.O. 2018)

Let $p, q \geq 5$ be prime and $N = pq$. If

$$f(\tau) = \prod_{0<\delta|N} \eta^{r_\delta}(\delta\tau)$$

is a weakly holomorphic modular form, then all of its orders of vanishing are congruent modulo $G/2h$, where $G = \gcd(p-1, q-1)$ and $h = \frac{1}{2}\gcd(24, p-1, q-1)$. Furthermore, the sum of the orders of vanishing equals

$$\frac{k(p+1)(q+1)}{12}.$$

Method

- Generate η-quotients which are elements of $S_2(\Gamma_0(N))$.
  - Calculate the distinct partitions of the sum of the orders of vanishings, $\frac{k(p+1)(q+1)}{12}$, and rearrangements.
  - Calculate r_1, r_p, r_q, and r_N.
  - Compute the q-expansions of the η's with the r_δ from above.
  - Check for linear independence.
- Attempt to construct a basis for $S_2(\Gamma_0(N))$ using these η-quotients.

Once a basis of η-quotients has been generated for $S_2(\Gamma_0(N))$, it is simple to express $f(\tau)$ in terms of the basis.

Result

Theorem 5 (A.H.O. 2018)

The elliptic curves with conductor $N = 35$ correspond to the following linear combination of η-quotients:

Conductor N	η - *quotient* $f(z)$
35	$\eta(\tau)^2\eta(35\tau)^2 + \eta(5\tau)^2\eta(7\tau)^2$

- Let N=35 and fix k=2. Then $\dim_{\mathbb{C}} S_2(\Gamma_0(35))=3$.
- The sum of the vanishings is 8, with 35 distinct rearrangements of the 5 distinct partitions.

For example, $M_2(\Gamma_0(22))$ has dimension 5, but contains only 4 linearly independent η-quotients:

$$\frac{\eta(2\tau)^4\eta(22\tau)^4}{\eta(\tau)^2\eta(11\tau)^2}, \eta(\tau)^2\eta(11\tau)^2, \eta(2\tau)^2\eta(22\tau)^2, \text{ and } \frac{\eta(\tau)^4\eta(11\tau)^4}{\eta(2\tau)^2\eta(22\tau)^2}.$$

A fifth basis element is $f(\tau) = q^4 + q^6 + q^8 + ...$ cannot be expressed as a linear combination of the η-quotients from above.

However, if $g(\tau) = \frac{\eta(22\tau)^{22}\eta(\tau)}{\eta(2\tau)^2\eta(11\tau)^{11}}$, then $f(\tau)g(\tau) \in M_{12}(\Gamma_0(22))$ and every holomorphic modular form in $M_{12}(\Gamma_0(22))$ is a linear combination of (holomorphic) η-quotients.

Extend

If we do not have enough η-quotients in our $S_2(\Gamma_0(N))$ space, then we begin to look for η's of higher weight (i.e. in $S_4(\Gamma_0(N))$, $S_6(\Gamma_0(N))$,...), similar to the method Rouse and Webb used in [3].

- Let $f_1, f_2, ..., f_n \in S_2(\Gamma_0(N))$, such that $n < \dim_{\mathbb{C}} S_2(\Gamma_0(N))$. Let $g \in S_2(\Gamma_0(N))$ be attached to elliptic curve E.
- Let $h_1, h_2, ..., h_i \in S_2(\Gamma_0(N))$ such that $\{f_1, f_2, ..., f_n, h_1, h_2, ..., h_i\}$ forms a basis for $S_2(\Gamma_0(N))$.
- Let k=4 and compute the η-quotients in $S_4(\Gamma_0(N))$. Let $a \in S_4(\Gamma_0(N))$ such that a is an arbitrary η-quotient.
- We then compute $ah_1, ah_2, ..., ah_i$, and know that $ah_1, ah_2, ..., ah_i \in S_6(\Gamma_0(N))$.
- Let $\beta = \{m_1, m_2, ..., m_{dim_{\mathbb{C}} S_6(\Gamma_0(N))}\}$ be a basis for $S_6(\Gamma_0(N))$.
- Write $ah_1, ah_2, ..., ah_i$, as a linear combination of the η-quotients in β.

Theorem

Theorem 6 (A.H.O. 2018)

The elliptic curves of conductor $N=55$ can be written as a linear combination of weakly holomorphic η-quotients.

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