Algorithmic investigation of substitution tilings and their associated graph Laplacians

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Background Information

Aperiodic Tiling

- An aperiodic tiling is a tiling of a plane that does not form repeating patterns.
- Some aperiodic tilings can be formed by applying inflate-and-subdivide (substitution) rules to an initial tile.
- 1-dimensional tilings are better understood than 2-dimensional tilings.

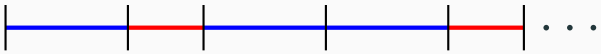


Figure 1: Fibonacci tiling

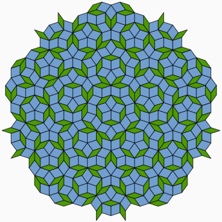
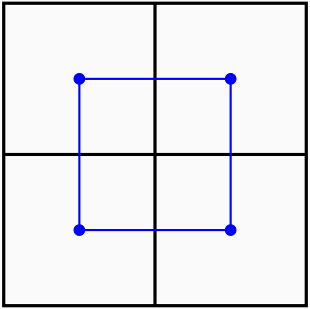


Figure 2: Penrose tiling
https://commons.wikimedia.org/wiki/File:Penrose_Tiling_(Rhombi).svg

Dual Graph

The dual graph G of a tiling has a node for each tile and an edge connect each pair of tiles that share an edge.



Graph Laplacian

The Laplacian matrix L of a graph is a matrix containing information about the structure of the graph.

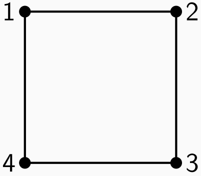


Figure 3: A basic graph.

(1,2)(2,3)(3,4)(4,1)
(2,1)(3,2)(4,3)(1,4)

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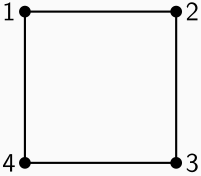

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$$D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

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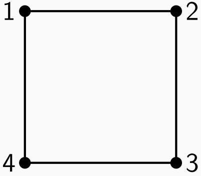

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$$D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

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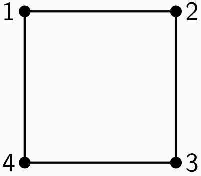

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$$L = D - A = \begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix}$$

Goal and Purpose

- Our goal: find a method to generate the Laplacian of substitution tilings in a 2-dimensional way based on their inflate-and-subdivide rules.

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- There exist 1-dimensional methods, but can we do it 2-dimensionally? Yes!

Chair Tiling

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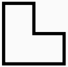
The **Chair Tiling** is an aperiodic tiling consisting of a single tile.

H_0



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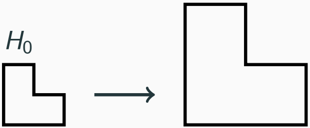

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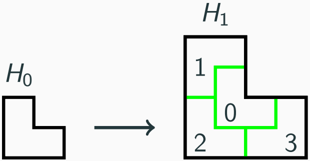

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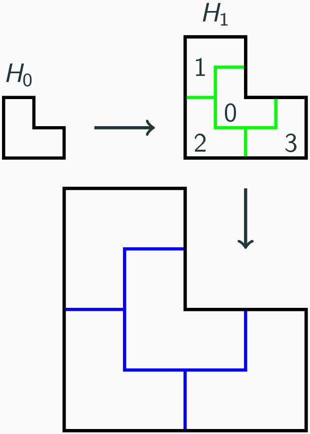

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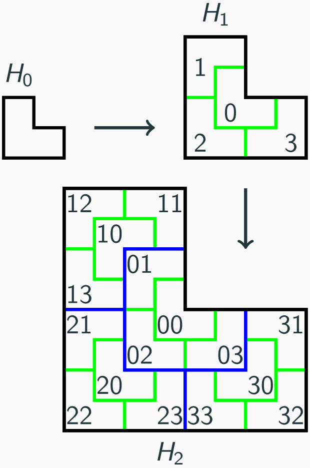
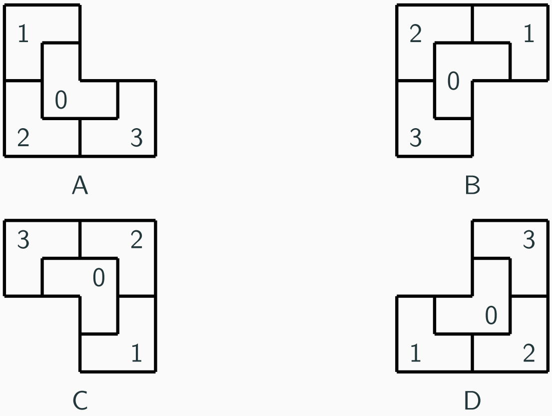
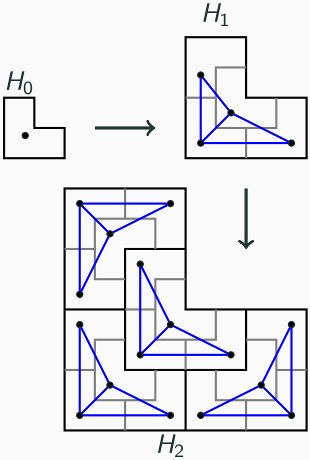


Rotations

There exist four different rotations of a sub-tiling.



Dual Graph



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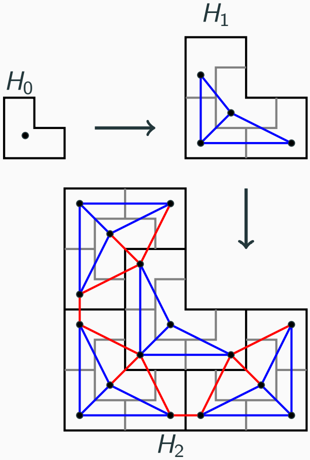
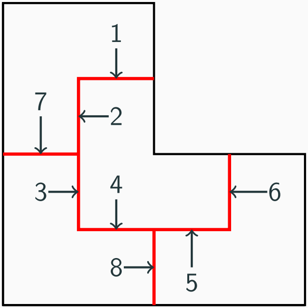
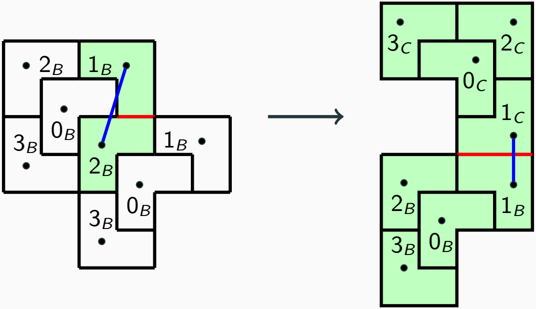


Quadrant Separations

We look at the substitution for each of the 8 line segments separating the quadrants.



Example



This case corresponds to the following rule in the 2-dimensional substitution.

- $(2_B, 1_B) \mapsto (1_B, 1_C)$ from line 1.

Line 1 Substitution

Line 1

North side	South side
$0_B \mapsto 0_B 1_B$	$2_B \mapsto 2_B 1_B$
$1_B \mapsto 1_C$	$1_B \mapsto 3_C 2_C$
$1_C \mapsto 1_D 2_D$	$3_C \mapsto 2_B 1_B$
$1_D \mapsto 2_A 3_A$	$2_C \mapsto 3_C 2_C$
$2_D \mapsto 1_D 2_D$	
$2_A \mapsto 2_A 3_A$	
$3_A \mapsto 1_D 2_D$	

Paired substitution

We pair the substitutions for either side of a line to create 2-d substitutions.

Line 1

$$
\begin{aligned}
(2_B, 0_B) &\mapsto (2_B, 0_B)(2_B, 1_B) \\
(2_B, 1_B) &\mapsto (1_B, 1_C) \\
(1_B, 1_C) &\mapsto (3_C, 1_D)(2_C, 2_D) \\
(3_C, 1_D) &\mapsto (1_B, 3_A)(2_B, 2_A) \\
(2_C, 2_D) &\mapsto (3_C, 1_D)(2_C, 2_D) \\
(1_B, 3_A) &\mapsto (3_C, 1_D)(2_C, 2_D) \\
(2_B, 2_A) &\mapsto (2_B, 2_A)(1_B, 3_A)
\end{aligned}
$$

Pinwheel Tiling

Pinwheel Substitution

- Defined by Radin and Conway
- Infinite orientations, so look at shapes created
- 5 quint-ants, so number in base 5
- For next iteration:
  - Inflate each tile by $\sqrt{5}$
  - Divide into copy of T_1

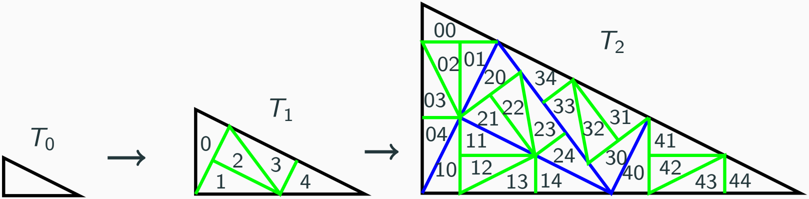
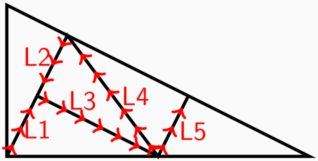


Figure 4: Pinwheel Tiling

Lines with New Edges

- 5 lines
- Direction will be important when finding 1-D substitution
- Direction doesn't affect edges
    - This way makes different lines have the same rules



2-D Substitution Across Lines

- Focus on shapes made across 5 lines
- Example with line between quint-ant 2 and quint-ant 3
- Split into two so we can see all the shapes
  - Middle tiles are part of 2 shapes because of adjacency to 2 tiles

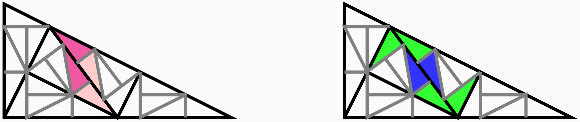
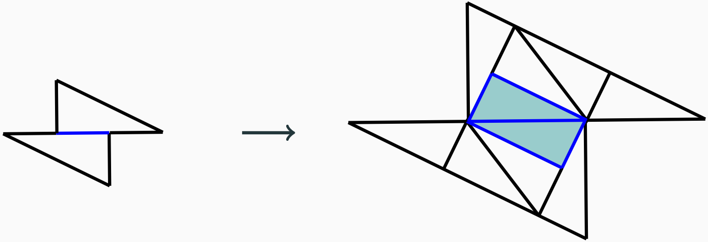


Figure 5: Shapes Formed between Quint-ants 2 and 3

2-D Substitution

- Blue line is where two tiles meet between quint-ants
- Want to know what shape will be on line after one iteration
- Use this process for all 7 of shapes that are created



2-D Substitution

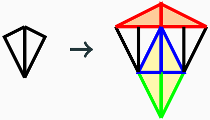


Figure 6: Forward Kite (A) and Backward Kite (B)

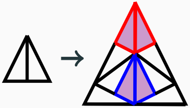


Figure 7: Forward Acute (C) and Backward Acute (D)

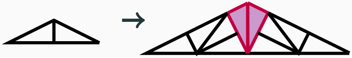


Figure 8: Forward Obtuse (E) and Backward Obtuse (F)

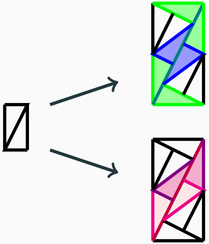
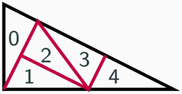


Figure 9: Rectangle (G)

1-D Substitution Naming

- Number based on the position inside the T_1 tiling that the tile is in
  - Last digit when in base 5
- Letter to represent the shape it creates across the edge, as well as direction
- i.e. 1_A if it's in the one spot, and a part of a forward kite



Quint-ant 0

$$1_B \mapsto 0_F\,3_D\,4_C \mapsto 0_B\,4_A\,1_A\,1_B\,4_B \mapsto 0_F\,3_D\,4_C\,4_D\,3_C\,0_E\,4_D\,3_C\,0_E\,3_D\,4_C\,0_F\,3_D\,4_C$$

$$0_B \mapsto 0_F\,3_D\,4_C \mapsto 0_B\,4_A\,1_A\,1_B\,4_B \mapsto 0_F\,3_D\,4_C\,4_D\,3_C\,0_E\,4_D\,3_C\,0_E\,3_D\,4_C\,0_F\,3_D\,4_C$$

Quint-ant 2

Figure 10: Line 2

Paired 1-D Substitution

Line 2

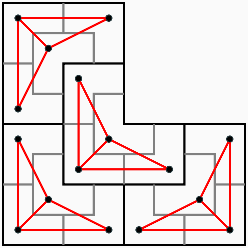
$(0_B, 1_B) \mapsto (0_F, 0_F)(3_D, 3_D), (4_C, 4_C)$
$(0_F, 0_F) \mapsto (0_B, 0_B)$
$(3_D, 3_D) \mapsto (4_A, 4_A)(1_A, 1_A)$
$(4_C, 4_C) \mapsto (1_B, 1_B)(4_B, 4_B)$
$(0_B, 0_B) \mapsto (0_F, 0_F)(3_D, 3_D)(4_C, 4_C)$
$(4_A, 4_A) \mapsto (4_D, 4_D)(3_C, 3_C)(0_E, 0_E)$
$(1_A, 4_A) \mapsto (4_D, 4_D)(3_C, 3_C)(0_E, 0_E)$
$(1_B, 1_B) \mapsto (0_F, 0_F)(3_D, 3_D)(4_C, 4_C)$
$(4_B, 4_B) \mapsto (0_F, 0_F)(3_D, 3_D)(4_C, 4_C)$
$(4_D, 4_D) \mapsto (4_A, 4_A)(1_A, 1_A)$
$(3_C, 3_C) \mapsto (1_B, 1_B)(4_B, 4_B)$
$(0_E, 0_E) \mapsto (0_A, 0_A)$
$(0_A, 0_A) \mapsto (4_D, 4_D)(3_C, 3_C)(0_E, 0_E)$

The Algorithm

$L_i = L_i^* + L_i'$

Our algorithm creates two intermediate Laplacians and adds them together to get the final Laplacian.

- L_i^* is a block matrix with four copies L_{i-1} on the diagonal blocks.
- L_i', is filled in using the pairs created from the paired substitutions.
- L_i is the Laplacian of the full connected graph.



L_2^*

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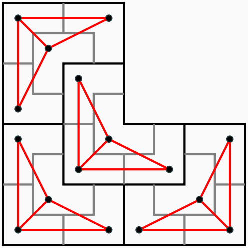

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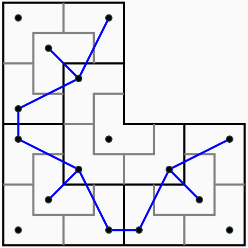

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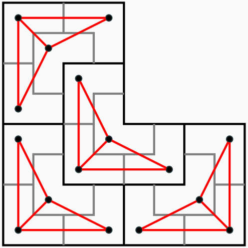

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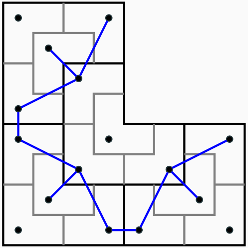

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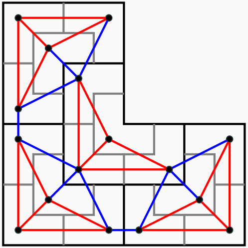

L_2

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