From Ideal Polyhedra to Fundamental Domains in $\mathbb{H}^3$

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Overview

Research Goal: to connect the geometric and topological properties of hyperbolic manifolds with the algebraic properties of their associated Kleinian groups

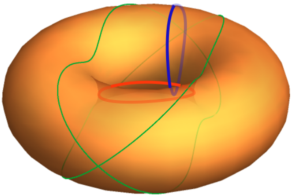
- Background on fundamental domains with a familiar example
- Important result about number of edge classes for a given abstract polyhedron
- Consequences of the theorem:
  - Classify all fundamental domains on the cube with torsion free groups
  - Results for more general fundamental domains

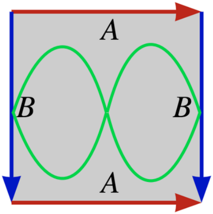
A Familiar Example

Definition (Hyperbolic Fundamental Domain)

A region that disjointly tiles hyperbolic space under the action of a Kleinian group (i.e. a discrete subgroup of isometries of $PSL(2, \mathbb{C})$).

Recall the torus and its fundamental domain in $\mathbb{R}^2$:





- Key: Tiling induces a set of identifications between the edges and also vertices of the domain

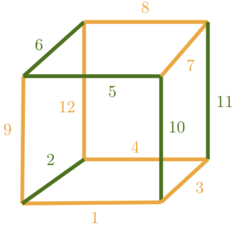
Hyperbolic Fundamental Domains

Generalize from torus example:

- Tiling induces face pairings, edge classes, and vertex identifications
- Impose the following conditions
  - Associated groups are *torsion-free*
  - Fundamental domains are *ideal* and *polyhedral*
- Result:
  - Obtain *smooth manifolds*
  - All edges in a class have interior *dihedral angles* summing to 2π
  - Can invoke useful results due to Rivin

Key equations relating exterior dihedral angles

- Rivin gave a characterization of when a given abstract polyhedron with prescribed exterior dihedral angles can be realized as a convex ideal polyhedron in $\mathbb{H}^3$
  - Sum of exterior dihedral angles for all edges incident to a given vertex is 2π
- Torsion-free condition $\implies$ angle sum of interior dihedral angles in an edge class of a fundamental domain is 2π



$x_1 = \frac{3\pi}{5}$ $x_5 = \frac{3\pi}{5}$ $x_9 = \frac{4\pi}{5}$

$x_2 = \frac{3\pi}{5}$ $x_6 = \frac{3\pi}{5}$ $x_{10} = \frac{3\pi}{5}$

$x_3 = \frac{4\pi}{5}$ $x_7 = \frac{4\pi}{5}$ $x_{11} = \frac{3\pi}{5}$

$x_4 = \frac{3\pi}{5}$ $x_8 = \frac{3\pi}{5}$ $x_{12} = \frac{4\pi}{5}$

My Key Theorem

Theorem (H.)

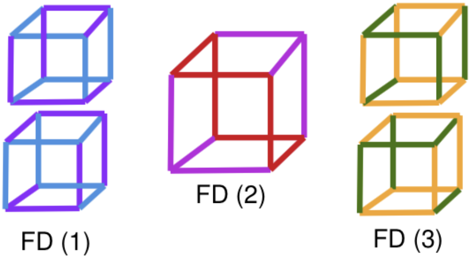
A polyhedron with $\overline{E}$ edges and $\overline{V}$ vertices must have $E = \dfrac{\overline{E} - \overline{V}}{2}$ edge classes.

Polyhedron	Vertices	Edges	Edge Classes in FD
Tetrahedron	4	6	1
Cube	8	12	2
Octahedron	6	12	3
Dodecahedron	20	30	5
Icosahedron	12	30	9

- Combining Theorem with the Euler characteristic equations for the abstract polyhedron and quotient manifold:
  - The number of vertex classes in the fundamental domain is equal to the Euler characteristic of the quotient space

Classifying Fundamental Domains on the Cube

- From theorem: must have 2 edge classes
- Show that edge classes must be broken down 5-7 or 6-6 to satisfy torsion-free and convexity conditions



Extending to more general polyhedra

- Classification of FDs on the cube was made significantly easier by knowing edge classes have at least 5 edges
  - Under what conditions do there need to be $>$ 3 edges per class?
  - How does the structure of the associated group affect the size of edge classes?

Background on Structure of Kleinian Groups

- Tiling of $\mathbb{H}^3$ induces face pairings and edge class partitions
  - Group elements that pair faces are *generators* of the group
  - Sequences of generators that traverse each edge class are *relators* of the group

Below is an example of such a group:

$$G = \langle A, B, C \mid AB^{-1}CA^{-1}B^{-1}C^{-1}, ABC^{-1}A^{-1}BC \rangle$$

A: Front$\longrightarrow$Back with $\frac{\pi}{2}$ clockwise twist on back interior face

B: Left$\longrightarrow$Right with $\frac{\pi}{2}$ clockwise twist on right interior face

C: Top$\longrightarrow$Bottom with $\frac{\pi}{2}$ clockwise twist on bottom interior face

Results for General Polyhedra

Proposition (H.)

If no pair of generators in the group corresponding to the fundamental domain commute with each other, then the fundamental domain does not have an edge class of size 3.

Corollary (H.)

If the group corresponding to a fundamental domain does not have any generators that commute, then $\overline{E} \leq 2\overline{V}$, where $\overline{E}, \overline{V}$ are the number of edges and vertices, resp., in the abstract polyhedron.

- Note that the icosahedron violates the inequality in the corollary and therefore a FD on the icosahedron must have commuting generators!

Possible Next Steps and Conclusions

- Some ideas for moving forward:
  - A similar characterization for the octahedron
  - Considering groups with non-trivial elements of finite order by introducing the additional equation

$$x_i = \frac{2\pi}{k}, k \in \mathbb{Z},$$

where x_i is the exterior dihedral angle along a certain edge

Conclusions and Acknowledgements

- Main Take-aways:
  - Algebraic properties of the associated group yield information about the topology and geometry of the associated polyhedron
  - These results help determine when a given polyhedron with prescribed dihedral angles can be a fundamental domain

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Bibiliography

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Heck, Laurel. *From Convex Ideal Polyhedra to Fundamental Domains in* $\mathbb{H}^3$. Submitted for publication to Minnesota Journal for Undergraduate Mathematics.

Image: https://mathoverflow.net/questions/219052/area-of-square-to-wrap-a-torus